New classes of exponentially general nonconvex variational inequalities

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ABSTRACT

In this paper, some new classes of exponentially general nonconvex variational inequalities are introduced and investigated. Several special cases are discussed as applications of these nonconvex variational inequalities. Projection technique is used to establish the equivalence between the non- covex variational inequalities and fixed point problem. This equivalent formulation is used to discuss the existence of the solution. Several inertial type methods are suggested and analyzed for solving ex- ponentially general nonconvex variational inequalities. using the technique of the projection operator and dynamical systems. Convergence analysis of the iterative methods is analyzed under suitable and appropriate weak conditions. In this sense, our result can be viewed as improvement and refinement of the previously known results. Our methods of proof are very simple as compared with other techniques.

1. Introduction

Variational inequalities theory, which was introduced by Stampacchia [1], can be viewed as a natural generalization and extension of the variational principles. It is tool of great power that can be applied to a wide variety of problems, which arise in almost all branches of pure, applied, physical, regional and engineering sciences. During this period, variational inequalities have played an important, fundamental and significant part as a unifying influence and as a guide in the mathematical interpretation of many physical phenomena. In fact, it has been shown that the variational inequalities provide the most natural, direct, simple and efficient framework for the general treatment of wide range of problems. Variational inequalities have been extended and generalized in several directions for studying a wide class of equilibrium problems arising in financial, economics, transportation, elasticity, optimization, pure and applied sciences, see [2], [3], [4]–[7], [8], [9]–[22], [23]–[36], [37], [38], [1], [39], [40] and the references therein. An im- portant and useful generalization of variational inequalities is called the general(Noor) variational inequality introduced by Noor [17] in 1988, which enables us to study the odd-order and nonsymmetric problems arising in physical oceanography, engineering and mathematical sciences a unified framework. See also [15], [11], [18]–[22], [13], [35]–[37] for the applications of general variational inequalities and related optimization problems. It is worth mentioning that almost all the results regarding the existence and iterative schemes for variational inequalities, which have been investigated and considered in the classical convexity. This is because all the techniques are based on the properties of the projection operator over convex sets, which may not hold in general, when the sets are nonconvex. Bounkhel at al. [2] and Noor [11]–[14] introduced and considered some new classes of variational inequalities, which are called the nonconvex variational inequality in

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conjunction with the uniformly prox-regular sets, which are nonconvex and include the convex sets as a special case.

Related to the general nonconvex variational inequalities, we have the problem of solving the nonconvex Wiener-Hopf equations. It is well known that the general non- convex variational inequalities are equivalent to the Wiener-Hopf equations. This alter- native equivalent formulation is more general and flexible than the projection operator technique. This alternative equivalent formulation has been used to suggest and ana- lyze a number of iterative methods for solving the nonconvex variational inequalities. Dupuis and Nagurney [5] introduced and studied the projected dynamical systems associated with variational inequalities using the equivalent fixed point formulation. The novel feature of the projected dynamical system is that the its set of stationary points corresponds to the set of the corresponding set of the solutions of the variation- al inequality problem. Thus the equilibrium and nonlinear programming problems, which can be formulated in the setting of the variational inequalities, can now be studied in the more general framework of the dynamical systems. For the applica- tions, sensitivity analysis, stability analysis and other aspects of dynamical systems, see [5], [10], [21], [29]–[31], [35], [5], [40].

It is known that the accurate inequalities can be derived using the algorithmically convex functions. Exponentially convex(concave) functions are closely related to the log-convex(concave) function. The origin of exponentially convex functions can be traced back to Bernstein [41]. Avriel [42] introduced and studied the concept of r-convex functions. Exponentially convex functions have important applications in information theory, big data analysis, machine learning and statistic, see [43], [44], [42], [41], [37], [45] and the references therein. Noor et al [46]–[49], [31], [32], [35] considered the concept of exponentially convex functions and discussed the basic their properties. It is worth mentioning that these exponentially convex functions considered by Noor et al [17] are distinctly differ- ent from the exponentially convex functions considered and studied by Bernstein [41]. It have been shown that the exponentially functions enjoy the same interesting properties which convex functions have. It have shown that the minimum of the differentiable exponentially convex functions can be characterized by the exponentially variational inequalities. Noor et al. [46], [48], [31], [32] introduced and studied some new classes of variational inequalities, which are called exponentially variational inequalities.

It is worth mentioning that the general variational inequalities, nonconvex varia- tional inequalities and exponentially variational inequalities are quite distinct classes of variational inequalities along with different applications. It is natural to develop a unified framework for these classes of variational inequalities. These facts and obser- vations motivated us to consider some new classes of exponentially general nonconvex variational inequalities(EGNVI).

In Section 2, we formulate the problem and discuss its special cases along with pre- liminaries results. Equivalence between EGNVI and fixed point problem is establish in Scetion3. This alternative equivalent formulation is used to discuss the existence of the solution. We shown that this alternative formulation played an important role is suggesting inertial type iterative methods for solving EGNVI. Convergence analy- sis is analyzed under suitable conditions. Wiener-Hopf technique is used to discuss some iterative for solving EGNVI in Section 4. In Section 5, we consider the second order initial value problem associated with EGNVI. Using the forward-backward finite difference scheme, we suggest and investigate some iterative methods for solving the EGNVI along with convergence criteria. Some special cases are also pointed as poten- tial applications of the obtained results. We have only considered theoretical aspects of the suggested methods. It is an interesting problem to implement these methods and to illustrate the efficiency. Comparison with other methods need further research efforts. The ideas and techniques of this paper may be extended for other classes of quasi variational inequalities and related optimization problems

2. Basic Concepts and Formulation

Let H be a real Hilbert space whose inner product and norm are denoted by $(.,.)$ and $\|.\|$ respectively. Let K be a nonempty closed convex set in H.

For given nonlinear operators $T, g: H \to H$ consider the problem of finding $u \in K$, where K is a convex set in H, such that:

$$(e^{TU}, g(v) - g(u)) \geq 0, \forall v \in K, \tag{2.1}$$

Which is called the exponentially general variational inequality, see [32]. We now show that the problem (2.1) arises as an optimality condition of the differen- tiable exponentially general convex functions. To be more precise, we recall the well known concepts and results for the sake of completeness and to convey the main ideas for the readers. which are mainly due to Noor and Noor [13], [14], [46], [32].

Definition 2.1 A set Kg in the Hilbert space *H* is said to be general convex set, if

$$(1-t)g(u) + tg(v) \in K_g, \forall u, v \in K_g, t \in [0,1]$$

It is important to emphasize that, if $g = I$, the identity operator, then the general convex set K_g = *K*, the classical convex set. Every convex set is a general convex set, but the converse is not true.

Definition (2.2) [46] A function F is said to be exponentially general convex function with to respect to an arbitrary function *g, if*

$$e^{f((1-t)g(u)+tg(v))} \leq (1-t)e^{f(g(u))} + te^{f(g(v))}, \forall u, v \in K, t \in [0,1]$$

We remark that Definition 2.6 can be rewritten in the following equivalent way,

Definition (2.3) [13], [46] A function *F* is said to be exponentially general convex func- tion with respect to an arbitrary function g, if

$$\mathrm{F}\left((1-\mathrm{t})g(u) + tg(v)\right) \leq \log\left[(1-t)e^{f(g(u))} + te^{F(g(v))}\right], \forall u, v \in K,\ t \in [0,1] \tag{2.2}$$

A function is called the exponentially general concave function F , if-*F* is exponentially general convex function.

If $g = I$, then Definition 2.7 reduces to

Definition (2.4) [50], [11] A function F is said to be exponentially convex function, if

$$e^{F((1-t)u+tv)} \leq (1-t)e^{F(u)} + te^{F(v)}, \forall u, v \in K,\ t \in [0,1]$$

which can be rewritten in the equivalent form

Definition (2.5) A function F is said to be exponentially convex function, if

$$F\left((1-t)u + tv\right) \leq \log\left[(1-t)e^{F(u)} + te^{F(v)}\right], \forall a, b \in K, t \in [0,1] \tag{2.3}$$

It is obvious that two concepts are equivalent. This equivalent have been used to discuss various aspects of the exponentially general convex functions. It is worth men- tioning that one can also deduce the concept of exponentially convex functions from r-convex functions, which were considered by Avriel [42] and Bernstein [41].

For the applications of the exponentially convex functions in the mathematical pro- gramming and information theory, see Antczak [44], Alirezaei et al. [43], Zhao et al. [45] and Pal et al [37]. For the applications of the exponentially concave function in the communication and information theory, we have the following example.

Example [1]: The error function

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2 dt,}$$

Becomes an exponentially concave function in the form erf (√x), $x \geq 0$, which de-scribes the bit/symbol error probability of communication systems depending on the square root of the underlying signal-to-noise ratio. This shows that the exponentially concave functions can play important part in communication theory and information theory.

We now consider the optimality condition for the differentiable exponentially general convex function, which is main focus of our nest result.

Theorem 2.1 Let F be a differentiable exponentially general convex function. Then u ∈ K is the minimum of the function F, if and only if, u ∈ K satisfies the inequality.

$$\left(e^{F(g(u))}F\left(g(u)\right), g(v)-g(u)\right) \geq 0, \forall u, v \in K \tag{2.4}$$

Proof. Let u ∈ k_g be a minimum of the function F. Then

$$F\left(g(u)\right) \leq F\left(g(v)\right), \forall v \in k_g$$

From which, we have

$$e^{F(g(u))} \leq e^{F(g(v))}, \forall v \in K_g \tag{2.5}$$

Since K is a general convex set, so, $\forall u, v \in K_{g,}\ t \in [0,1]$,

$$g(v_t) = (1-t)g(u) + tg(v) \in K_g$$

Taking g(v) = g(v_t) in (2.5), we have

$$0 \leq \lim_{t\to 0}\{\frac{e^{F\left(g(u)+t(g(v)-g(u))\right)} - e^{F(g(u))}}{t}\} = (e^{F(g(u))}F\left(g(u)\right), g(v)-g(u)), \tag{2.6}$$

which is the required (2.4)

Since F is differentiable exponentially general convex function, so

$$e^{F(g(v))} - e^{F(g(u))} \geq \lim_{t\to 0}\{\frac{e^{F(g(u)+t(g(v)-g(u)))} - e^{F(g(u))}}{t}$$

$$= (e^{F(g(u))}F\left(g(u)\right), g(v)-g(u)) \geq 0,$$

Which implies that

$$F\left(g(u)\right) \leq F\left(g(v)\right), \forall v \in K_g$$

This shows that u Kg is the minimum of the differentiable exponentially general convex function, the required result.

The inequality of the type ((2.4)) is called the exponentially general variational in- equality, which is a special case of (2.1).

The basic concepts and definitions used in this paper are exactly the same as in Noor [11], [13], [14]. Poliquin et al. [51] and Clarke et al. [3] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets.

Definition 2.6 [3], [51] The proximal normal cone of K at u ∈ H is given by

$$N_K^P(u) = \{\xi \in H: u \in P_K[u+\alpha\xi]\},$$

Where α > 0 is a constant and

$$P_K[u] = \{u^* \epsilon K: d_K(u) = \left|\left|u-u^*\right|\right|\}$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v\epsilon K} ||v-u||$$

The proximal normal cone $N_K^P(u)$ has the following charcterization.

Lemma 2.1. Let K be a nonempty, closed and convex subset in *H*. Then $\zeta \epsilon N_K^P(u)$.

If and only if, there erists a constant α > 0 such that

$$(\zeta, v-u) \leq \alpha||v-u||^2, \forall v \in K$$

Definition 2.7. The Clarke normal cone, denoted by $N_K^C(u)$ is defined as

$$N_K^C(u) = \overline{co}[N_K^P(u)],$$

Where $\overline{co}$ means the closure of the convex hull.

Clearly $N_K^P(u) \subset N_K^C(u)$, but the converse is not true. Note that $N_K^P(u)$ closed and convex, whereas $N_K^C(u)$ is convex, but may not be closed [3], [51].

Definition 2.8. For a given r ∈ (0,∞], a subset K_r is said to be normalized

uniformly r-prox-regular, if and only if, every nonzero proximal normal cone to K_r can be realized by an r-ball, that is, $\forall u \in K, and\ 0 \neq \xi \epsilon N_{K_r}^P(u)$, one has

$$((\frac{\xi}{||\xi||}, v-u) \leq (1/2r)||v-u||^2, \forall v \in K_r$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [3], [11], [12], [14], [31]. Obviously, for r = ∞, the uniformly prox-regularity of K_r is is equivalent to the convexity of K. This class of uniformly prox-regular sets have played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions. It is known that if K_r is a uniformly proxregular set, then the proximal normal cone $N_{K_r(u)}^P$ is closed as a set-valued mapping.

We now recall the well known proposition which summarizes some important properties of the uniformly prox-regular sets K_r.

Lemma 2.2. [3] Let *K* be a nonempty closed subset of *H*, r ∈ (0, ∞] and set $K_r = \{u \in H : d_k(u) < r\}. if\ K_r\ _{is\ uniformly\ prox-regular, then}$

(i). $\forall u \in K_{r,} P_{Kr}(u) \neq \emptyset$.

(ii). $\forall r \in (0, r), P_{kr}\ is\ Lipschitz\ continuous\ with\ constant\ \delta = \frac{r}{r-r^1}\ on\ K_{r^1}.$

We now consider the exponentially general nonconvex variational inequality(EGNVI). To be more precise, for given nonlinear operators T, g : H → R, we consider the problem of finding u ∈ K_r such that.

$$(e^{Tu}, g(v) - g(u) \geq 0, \forall v \in K_r, \tag{2.7}$$

Which is called the $exponentially\ general\ nonconvex\ variational\ inequality.$

Special caes. Some important special cases of the problem (2.7) are discussed.

(I), if $K_r^* = \{u \in H: (u, v) \geq 0, \forall v \in K_r\}$ is the polar cone of the nonlinear convex cone K_r, then the problem (2.7) is equivalent to finding u ∈ H, such that.

$$g(u) \in K_{r,} e^{Tu} \in K_r^*, \left(e^{Tu}, g(u)\right) = 0 \tag{2.8}$$

is called the exponentially general complementarity problem. For the applications and formulation of the complementarity problems in engineering and mathematical sciences, see [52], [7], [53], [17], [19], [21], [35].

(II). For $K_r = K$, the convex set K in H, the problem (2.7) collapse to find uıK

such that

$$\left(e^{Tu}, g(v) - g(u)\right) \geq 0, \forall v \in K, \tag{2.9}$$

which is called the exponentially general variational inequality, introduced and studied in [48], [32].

(III). We note that, if $e^{Tu} = N(u)$, where N is a nonlinear operator, then problem (2.7) is equivalent to finding u ∈ H : g(u) ∈ K such that.

$$\left(N(u), g(v) - g(u)\right) \geq 0, \forall v \in H: g(v) \in K, \tag{2.10}$$

which is known as the general variational inequality, introduced and studied by Noor [17] in 1988. It have shown that nonsymmterics, nonpositive and odd order obstacle boundary value problems can be studied in the unified framework of the general variational inequalities. For the applications, numerical methods, formulation and other aspects of the general variational inequalities (2.10), see [19], [21], [22], [13], [35] and the references therein.

(IV). If $g \equiv I$, the identity operator, then problem (2.7) is equivalent to finding

u ∈ K_r such that

$$(e^{Tu}, v - u) \geq 0, \forall v \in K_{r,} \tag{2.11}$$

which is called the exponentially variational inequality.

(V). If $K_r \equiv K$, the convex set in H, and $g \equiv I$, the identity operator, then problem (2.7) is equivalent to finding u ∈ K such that.

$$(e^{Tu,} v - u) \geq 0, \forall v \in K. \tag{2.12}$$

Inequality of type (2.12) is called the exponentially variational inequality.

(VI). If $e^{Tu} = S(u)$, where S is a nonlinear operator, then the problem (2.12) is equivalent to finding u ∈ K such that

$$(S(u), v - u) \geq 0, \forall v \in K. \tag{2.13}$$

is known as variational inequality, which was introduced and studied by Stampacchia [1] in potential theory. It turned out that a number of unrelated obstacle and and equilibrium problems arising in various branches of pure and applied sciences can be studied via variational inequalities, see [50], [2], [52], [4]–[7], [54], [9], [10], [15], [17], [19], [11]–[14], [21], [22], [46], [47], [23], [24], [29]–[32], [5], [22], [26], [28], [34], [35], [55], [56], [38], [57], [1], [39], [40] and the references therein.

If K_r is a nonconvex (uniformly prox-regular) set, then problem (2.7) is equivalent to finding u ∈ K_r such that

$$0 \in pe^{Tu} + g(u) - g(u) + \rho N^P_{K_r}(g(u)) \tag{2.14}$$

where $N^P_{K_r}\left(g(u)\right)$ (g(u)) denotes the normal cone of Kr at g(u) in the sense of nonconvex analysis. Problem (2.14) is called the general nonconvex variational inclusion problem associated with general nonconvex variational inequality (2.7). This equivalent formulation plays a crucial and basic part in this paper. We would like to point out this equivalent formulation allows us to use the projection operator technique for solvin the exponentially general nonconvex variational inequalities of the type (2.7).

Definition 2.9. *An operator $T : H \rightarrow H$ with respect to an arbitrary operator g is said*

(i). exponentially general monotone if

$$\left(e^{Tu}-e^{Tv}, g(u)-g(v)\right) \geq 0, \forall u, v \in H.$$

(ii). exponentially general pseudomonotone if

$$\left(e^{Tu}, g(v)-g(u)\right) \geq 0, implies \left(e^{Tu}, g(v)-g(u)\right) \geq 0, \forall u, v \in H.$$

It is well known that exponentially general monotonicity implies exponentially general pseudomonotonicity, but, the converse is not tru

3. Methods

In this section, we prove that the EGNVI are equivalent to the fixed point problems. This alternative equivalent formulation is used to discuss the existence results and propose several iterative methods for solving the EGNVI. Convergence analysis is also analyzed under suitable conditions. First of all, we establish the equivalence between ENGVI (2.7) and fixed point formulation. This is the main motivation of nest result.

Lemma 3.1. [58] u ∈ *H : g(u)* ∈ K_r is a solution of the exponentially general nonconvex variational inequality (2.7), if and only if, u ∈ K_r satisfies the relation

$$g(u)=P_{Kr}[g(u)-\rho e^{Tu}], \tag{3.1}$$

where P_{Kr} is the projection of *H* onto the uniformly prox-regular set $K_{r.}$

Proof. Let u ∈ $K_{r.}$ be a solution of (2.7). Then, for a constant ρ > 0,

$$\begin{aligned} 0 &\in g(u)+\rho N_{K_r}^{P}\left(g(u)\right)-\left(g(u)-\rho e^{Tu}\right) \\ &= (I+\rho N_{Kr}^{P})\left(g(u)\right)-\left(g(u)-\rho e^{Tu}\right) \\ &\Leftrightarrow \end{aligned}$$

$$g(u)=(I+\rho N_{K_r}^{P})^{-1}[g(u)-\rho e^{Tu}]$$

where we have used the well-known fact $P_{Kr} \equiv (I+N_{Kr}^{P})^{-1}$, the required results.

Lemma 3.1 implies that the exponentially general nonconvex variational inequality (2.7) is equivalent to the fixed point problem (3.1). This alternative equivalent formulation is very useful from the numerical and theoretical points of view. We rewrite the relation (3.1) in the following form

$$F(u)=u-g(u)+P_{Kr}[g(u)-\rho e^{Tu}], \tag{3.2}$$

which is used to study the existence of a solution of the exponentially general nonconvex variational inequality (refeq1.1).

We now study those conditions under which the exponentially general nonconvex variational inequality (2.7) has a solution and this is the main motivation of our next result.

Theorem 3.1. *Let P_{Kr} be the Lipschitz continuous operator with constant $\delta = \frac{r}{r-r}$ Let T, g be strongly exponentially general monotone with constants $\alpha > 0, \sigma$ 0 and Lipschitz continuous with constants $\beta > 0, \delta$ 0, respectively. If there exists a constant $\rho > 0$ such that*

$$\left|\rho-\frac{\alpha}{\beta^2}\right| < \frac{\sqrt{\delta^2\alpha^2-\beta^2(1-(1+\delta)k))^2)}}{\delta\beta^2}, \tag{3.3}$$

$$\delta\alpha > \beta\sqrt{k(1+\delta)\left(2-k(1+\delta)\right)},\ \kappa < \frac{1}{1+\delta'} \tag{3.4}$$

Where

$$k = \sqrt{1 - 2\sigma + \delta^2}, \tag{3.5}$$

then there exists a solution of the exponentially general nonconvex variational inequality (2.7).

Proof. From Lemma 3.1, it follows that problems (3.1) and (2.7) are equivalent.

Thus it is enough to show that the map F(u), defined by (3.2), has a fixed point. For all u 6= v ∈ K_r, we have

$$||F(u) - F(v)|| = ||u - v - \big(g(u) - g(v)\big)|| + ||P_{Kr}[g(u) - \rho e^{Tu}] - P_{Kr}[g(v) - \rho e^{Tu}||| \tag{3.6}$$

$$\leq ||u - v - \big(g(u) - g(v)\big)|| + \delta||u - v - \rho(e^{Tu} - e^{Tu})||$$
$$+\delta||u - v - \big(g(u) - g(v)\big)||,$$

where we have used the fact that the operator P_{Kr} is a Lipschitz continuous operator with constant δ.

Since the operator T is strongly exponentially general monotone with constant α > 0 and Lipschitz continuous with constant β > 0, it follows that

$$||u - v - \rho(e^{Tu} - e^{Tu})||^2 \leq \big|\big|u - v||^2 - 2\rho(e^{Tu} - e^{Tu}, u - v) + \rho^2\big|\big|e^{Tu} - e^{Tv}||^2$$
$$\leq (1 - 2\rho\alpha + \rho^2\beta^2)||u - v||^2 \tag{3.7}$$

In a similar way, we have

$$||u - v - (g(u) - g(v)|| \leq \sqrt{1 - 2\sigma + \delta^2}||u - v| = k||u - v||, \tag{3.8}$$

where σ > 0 is the strongly monotonicity constant and δ > 0 is the Lipschitz continuity constant of the operator g respectively.

From (3.6), (3.7) and (3.8), we have

$$||F(u) - F(v)|| \leq \{k + \delta\big\{k + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2)}\big\}||u - v||$$
$$= \theta||u - v||,$$

Where

$$\theta = k + \delta\big\{\sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\big\} \tag{3.9}$$

From (3.3) and (3.4), it follows that θ < 1, which implies that the map F(u) defined by (3.2), has a fixed point, which is a unique solution of (2.7).

This fixed point formulation (3.1) is used to suggest the following iterative method for solving the exponentially general nonconvex variational inequality (2.7). Using the fixed point formulation (3.1), we suggest and analyze the several iterative methods for solving the exponentially general nonconvex variational inequality (2.7). The equation (3.1)can be rewritten equivalently as

$$g(u) = P_{Kr}[(1 - \alpha)g(u) + \alpha g(u) - \rho e^{Tu}]$$
$$= P_{Kr}[g\big((1 - \alpha)u + \alpha u\big) - \rho e^{T((1\xi)u + \xi u)}] \tag{3.10}$$

where α, ξ ∈ [0, 1] are constants. This equivalent formulation is used to suggest and analyzed the following iterative methods for solving the problem (2.7).

Algorithm 3.1. *For a given* $u0 \in H$*, find the approximate solution* u_{n+1} *by the iterative*

Schemes

$$g(u_{n+1}) = P_{Kr}[(1-\alpha)g(u_n) + \alpha g(u_{n+}) - \rho e^{T_{uv+1}}]$$

$$= P_{Kr}[g\big((1-\alpha)u_n + \alpha u_{n+1}\big) - \rho e^{T\left((1-\xi)u_n+\xi u_{n+1}\right)}]$$

For α = 0, ξ = 0, Algorithm 3.1 reduces to

Algorithm 3.2. *For a given u0 ∈H, find the approximate solution un+1 by the iterative schemes*

$$g(u_{n+1}) = P_{Kr}\big[g(u_n) - \rho e^{T(u_{n+1})}\big], n = 0,1 \ldots.$$

Which is called the explicit iterative method.

For $\alpha = 0,\ \xi = 1$, Algorithm 3.1 reduces to.

Algorithm 3.3 *For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = P_{K_r}\big[g(u_n) - \rho e^{T(u_{n+1})}\big],\ n = 0,1,\ldots \tag{3.11}$$

Algorithm 3.3 is an implicit iterative method for solving the exponentially general nonconvex variational inequalities (2.7).

To implement the Algorithm 3.3, we use the predictor-corrector technique. We use Algorithm 3.2 as predictor and Algorithm 3.3 as a corrector to obtain the following predictor-corrector method for solving the exponentially general nonconvex variational inequality (2.7).

Algorithm 3.4. *For a given u_0 ∈H, find the approximate solution u_{n+1} by the iterative schemes*

$$g(y_n) = P_{Kr}[g(u_n) - \rho e^{T_{un}}]$$

$$g(u_{n+1}) = P_{Kr}[g(u_n) - \rho e^{T_{yn}}], n = 0,1,\ldots.$$

Algorithm 3.4 is known as the extragradient method in the sense of Korpelevich [8]. It is obvious that the implicit method (Algorithm 3.3) and the extragradient method (Algorithm 3.4) are equivalent. We use this equivalent formulation to study the convergence analysis of Algorithm 3.3 and this is the motivation of our nest result.

We now consider the convergence analysis of Algorithm 3.3 and this is the main motivation of our next result.

Theorem 3.2. *Let u ∈H : g(u) ∈$K_r$be a solution of (2.7) and let u_{n+1} be the approximate solution obtained from Algorithm 3.3. If the operator T is exponentially general pseudomonotone, then*

$$||g(u) - g(u_{n+1})||^2 \le ||g(u) - g(u_n)||^2 - ||g(u_{n+1}) - g(u_n)||^2 \tag{3.12}$$

Proof. Let $u \in H : g(u) \in K_r$ be solution of (2.7). Then, using the exponentially general pseudomonotonicity of T, we have

$$\big(e^{Tu}, g(v) - g(u)\big) \ge 0, \forall v \in K_r \tag{3.13}$$

Take $v = u_{n+1}$ in (5.10), we have

$$(e^{T(u_{n+1})}, g(u_{n+1}) - g(u)) \ge 0. \tag{3.14}$$

Using Lemma 2.1, equation (3.11) can be written as

$$\left(\rho e^{T(u_{n+1})} + g(u_{n+1}) - g(u_{n+1})\right) \geq 0, \forall v \in H; g(v) \in K_{r.} \tag{3.15}$$

Taking v = u in (5.11), we have

$$(g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})) \geq 0 \tag{3.16}$$

From (5.13) and (5.11), we have

$$(g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1})) \geq 0,$$

Which implies that

$$||g(u) - g(u_{n+1})||^2 \leq ||g(u) - g(u_n)||^2 - ||g(u_{n+1}) - g(u_n)||^2$$

The required results (3.12).

Theorem 3.3. *Let u $\in H$: g(u) $\in K_r$ be a solution of (2.7) and let u_{n+1} be the approximate solution obtained from Algorithm 3.3. If H is a finite dimensional space and g^{-1} exists, then $lim_{n \to \propto} u_n$ = u.*

Proof. Let $\bar{u}$ $\in\in H$: $g(\bar{u})$ $\in K_r$ be a solution of (2.7). Then, the sequencesi $\{||g(u_n) - g(\bar{u})||\}$ is nonincreasing and bounded and

$$\sum_{n=0}^{\infty} ||g(u_{n+1} - g(u_n)||^2 \leq ||g(u_0) - g(\bar{u})||^2,$$

which implies

$$\lim_{n \to \infty} \left||u_{n+1} - u_n|\right| = 0, \tag{3.17}$$

where we have used the fact that g^{-1} exist.

Let $\bar{u}$ be a cluster point of $\{u_n\}$; there exists a subsequence $\{u_{ni}\}$ such that $\{u_{ni}\}$ converges to $\bar{u}$. Replacing u_{n+1} by u_{ni} in n (5.11) and taking the limits and using (3.17), we have.

$$\left(e^{T\bar{u}}, g(v) - g(\bar{u})\right) \geq 0, \forall v \in H: g(v) \in K_r$$

This shows that $\bar{u}$ $\in H$: *$g(\bar{u})$* $\in K_r$ solves the exponentially general nonconvex variational inequality (2.7) and

$$||g(u_{n+1}) - g(\bar{u})||^2 \leq ||g(u_n) - g(\bar{u})||^2$$

which implies that the sequence $\{u_n\}$ has a unique cluster point and limn→∞ u_n = $\bar{u}$, is the solution of (2.7), the required result.

Remark 3.1. We also remark that, if the operator g is a nonexpanding operator [2], namely

$$||g(u) - g(v) \geq ||u - v|, \forall u, v \in H.$$

Then Theorem 3.3 continues to hold.

For α =1, $\xi = 1$, Algorithm 3.1 collapes to

Algorithm 3.5. *For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes*

$$g(u_{n+1}) = P_{Kr}\left[g(u_{n+1}) - \rho e^{T(u_{n+1})}\right], n = 0,1, \ldots$$

Algorithm 3.5 is an implicit iterative method for solving the exponentially general nonconvex variational inequalities (2.7).

To implement the Algorithm 3.5, we use the predictor-corrector technique. We use Algorithm 3.2 as predictor and Algorithm 3.5 as a corrector to obtain the following predictor-corrector method for solving the exponentially general nonconvex variational inequality (2.7).

Algorithm 3.6. For a given $u_0 \in H$, find the approximate solution u_{n+1} by the iterative schemes

$$g(w_n) = P_{Kr}[g(u_n) - \rho e^{Tun}] \quad (3.18)$$

$$g(u_{n+1}) = P_{Kr}[g(w_n) - \rho e^{Twn}], n = 0,1, \dots \quad (3.19)$$

Algorithm 3.6 is known as the modified extragradient method in the sense of Noor [18], [19]. We would like to remark that this modified extragradient method is quite different than the extragradient method, which was suggested by Korpelevich [8]. Here we would like to point out that the implicit method (Algorithm 3.5) and the extragradient method (Algorithm 3.6) are equivalent. We use this equivalent to prove the convergence of the implicit projection method (Algorithm 3.5), which requires only the partially relaxed strongly monotonicity. We now consider the convergence analysis of Algorithm 3.5.

Theorem 3.4. *Let $u \in H : g(u) \in K_r$ be a solution of (2.7) and let un+1 be the approximate solution obtained from Algorithm 3.5. If the operator T is partially relaxed strongly exponentially general monotone with constant α > 0, then*

$$||g(u_{n+1}) - g(u)||^2 \le ||g(w_n) - g(u)||^2 - (1 - 2\alpha\rho)||g(u_{n+1}) - g(w_n)||^2 \quad (3.20)$$

$$||g(w_n) - g(u)||^2 \le ||g(u_n) - g(u)||^2 - (1 - 2\alpha\rho)||g(w_n) - g(u_n)||^2 \quad (3.21)$$

Proof. Let $u \in H : g(u) \in K_r$ be solution of (2.7). Then

$$\left(e^{Tu}, g(v) - g(u)\right) \ge 0, \forall v \in H : g(v) \in K_r \quad (3.22)$$

Take $v = w_n$ in (3.22), we have

$$(e^{Tu}, g(w_n) - g(u)) \ge 0. \quad (3.23)$$

Using Lemma 3.1, equation (3.18) can be written as

$$\left(\rho e^{Tun} + g(w_n) - g(u_n), g(v) - g(w_n)\right) \ge 0, \forall v \in H: g(v) \in K_r \quad (3.24)$$

Taking $v = u$ in (3.24) and using (3.23), we have

$$(g(w_n) - g(u_n), g(u) - g(w_n)) \ge \rho(e^{Tun} - e^{Tu}, g(w_n) - g(u))$$

$$\ge -\alpha\rho||g(u_n) - g(w_n)||^2 \quad (3.25)$$

since T is partially relaxed strongly exponentially general monotone with constant

α > 0.

From (3.25), we have

$$||g(u) - g(w_n)||^2 \le ||g(u) - g(u_n)||^2 - (1 - 2\alpha\rho)||g(w_n) - g(u_n)||^2$$

the required results (3.21).

Now taking $v = u_{n+1}$ in (3.22), we have 2.5

$$(T_u, g(u_{n+1}) - g(u)) \ge 0 \quad (3.26)$$

Using Lemma 3.1 , we rewrite (3.19) in the equivalent form as:

$$\left(\rho e^{Twn} + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1})\right) \ge 0, \forall v \in H : g(v) \in K_r \quad (3.27)$$

Taking $v = u$ in (3.27), we have

$$\left(\rho e^{Twn} + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1})\right) \geq 0 \tag{3.28}$$

From (3), (3.27) and using the partially relaxed strongly exponentially general monotonicity of T with constant α > 0, we have

$$||g(u_{n+1}) - g(u)||^2 \leq ||g(w_n) - g(u)||^2 - (1 - 2\alpha\rho)||g(u_{n+1}) - g(w_n)||^2,$$

The required results (3.20).

Theorem 3.5 *Let* $u \in H$ " $g(u) \in K_r$ *be a solution of* (2.7) *and let* u_{n+1} *be the approximate solution obtained from Algorithm 3.5. Let H be a finite dimensional space and* g^{-1} *exist. If* $0 < \rho < \frac{1}{2\alpha}$, *then* $\lim_{n\to\infty} u_n = u$.

Proof. Let $\bar{u} \in H : g(\bar{u}) \in K_r$ be a solution of (2.7). Then, the sequences $\{||g(u_n) - g(\bar{u})||\}$ is nonincreasing and bounded and.

$$\sum_{n=0}^{\infty} (1 - 2\alpha\rho)||g(u_{n+1}) - g(w_n)||^2$$

$$\sum_{n=0}^{\infty}(1 - 2\alpha\rho)||g(w_n) - g(u_n)||^2 \leq ||g(u_0) - g(u)||^2$$

Which implies

$$\lim_{n\to\infty}||g(u_{n+1}) - g(w_n)|| = 0$$

$$\lim_{n\to\infty}||g(w_n) - g(u_n)|| = 0$$

Thus

$$\lim_{n\to\infty}||g(u_{n+1}) - g(u_n)|| = \leq \lim_{n\to\infty}||g(u_{n+1}) - g(w_n)|| + \lim_{n\to\infty}||g(w_n) - g(u))|| = 0. \tag{3.29}$$

Since g^{-1} exists, it follows that

$$\lim_{n\to\infty}||u_{n+1}) - u_n|| = 0.$$

Let $\bar{u}$ be a cluster point of $\{u_n\}$; there exists a subsequence $\{u_{ni}\}$ such that $\{u_{ni}\}$ converges to $\bar{u}$. Replacing u_{n+1} by u_{ni} in (3.24) w_n by u_{ni} in (3.28) and taking the limits and using (3.30), we have

$$(e^{T\bar{u}}, g(v) - g(\bar{u})||^2 \leq ||g(u_n) - g(\bar{u})||^2,$$

which implies that the sequence $\{u_n\}$ has a unique cluster point and $\lim_{n\to\infty} u_n = \bar{u}$, is the solution of (2.7), the required result.

For $\alpha = \frac{1}{2}$ *and* $\alpha = \frac{1}{2}, \xi = \frac{1}{2}$. Algorithm 3.1 reduces to the following iterative methods

Algorithm 3.7 *For a given* $u_0 \in H$, *find the approximate solution* u_{n+1} *by the iterative schemes.*

$$g(u_{n+1}) = P_{Kr}\left[\frac{g(u_n)+g(u_{n+1})}{2} - \rho e^{Tu_{n+1}}\right], n = 0,1,2, \ldots$$

and

Algorithm 3.8. *For a given* $u_0 \in H$, *find the approximate solution* u_{n+1} *by the iterative schemes.*

$$g(u_{n+1}) = P_{Kr}\left[g\left(\frac{u_n+u_{n+1}}{2}\right) - \rho e^{T\left(\frac{un+un+1}{2}\right)}\right]$$

which are called midpoint implicit methods for solving the exponentially general nonconvex variational inequalities. Using the above techniques and ideas of this paper, one can consider the convergence analysis of these algorithms. We again rewrite the equation (3.1) in the equivalently as:

$$g(u) = P_{Kr}\left[g\left(((1-\lambda u) + \lambda u)\right) - \rho e^{T((1-\lambda u)+\lambda u))}\right]$$

$$= P_{Kr}\left[g(y) - \rho e^{T(y)}\right]; \quad (3.30)$$

Where $\lambda \in [0,1]$ is a constant and

$$y = (1-\lambda)u + \lambda u \quad (3.31)$$

This equivalent fixed point formulations is used to suggest the following iterative method

Algorithm 3.9. *For given $u_{o,}u_1 \in H$, find the approximate solution u_{n+1} by the iterative schemes.*

$$yn = (1-\lambda)u_n + \lambda u_{n-1}$$

$$g(u_{n+1}) = P_{Kr}\left[g(yn) - \rho e^{T(y_n)}\right],$$

which is known as the inertial implicit iterative method for solving the problem (3.1). In a similar way, we can propose the following method.

Algorithm 3.10. *For a given $u_o, u_1 \in H$, find the approximate solution u_{n+1} by the iterative schemes*

$$y_n = (1-\lambda)u_n + \lambda u_{n-1}$$

$$g(u_{n+1}) = P_{Kr}\left[g(u_n) - \rho e^{T(y_n)}\right],$$

This method is called the extragradient method of Korpelevich [8]. In this paper, we suggest and analyze the following two-step iterative method for solving the exponentially general nonconvex variational inequalities (2.7).

Algorithm 3.11. *For a given $u_o, \in K_r$ find the approximate solution u_{n+1} by the iterative schemes*

$$y_n = (1-\beta_n)u_n + \beta_n\{y_n - g(y_n) + P_{K_r}[g(u_n) - \rho T_{u_n}]\} \quad (3.32)$$

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{u_n - g(u_n) + P_{K_r}[g(y_n) - \rho T_{y_n}]\}, n = 0,1,2,\ldots \quad (3.33)$$

where $\alpha_{n,}\beta_n \in [0,10], \forall n \geq 0.$

Clearly for $\alpha_n = \beta_n = 1$, Algorithm 3.11 reduces to Algorithm 3.6. It is worth mentioning that, if r = ∞, then the nonconvex set K_r reduces to a convex set *K*. Consequently all the Algorithms collapse to the following algorithms for solving the general variational inequalities.

We now consider the convergence analysis of Algorithm 3.11 and this is the main motivation of our next result. In a similar way, one can consider the convergence criteria of other Algorithms.

Theorem 3.6. *Let P_{Kr}be the Lipschitz continuous operator with constant $\delta = \frac{r}{r-r}$. Let the operators $T, g: H \to H$be strongly exponentially general monotone with constans $\alpha > 0, \sigma > 0$ and Lipschitz continuous with constants with $\beta > 0, \delta > 0$, respectively. If (3.3), (3,4), (3.5) hold, $\alpha_n, \in [0,1], \forall n \geq 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, then the approximate solution u_n hold, $\alpha_n, \in [0,1], \forall n \geq 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, then the approximate solution u_n obtained from Algorithm 3.11 converges to a solution $u \in K_r$ satisfying the exponentially general nonconvex variational inequality (2.7).*

Proof. Let $u \in H : g(u) \in K_r$ be a solution of the exponentially general nonconvex variational inequality (2.7). Then, using Lemma 3.1, we have

$$u = (1-\alpha_n)u + \alpha_n\{u - g(u) + P_{Kr}[g(u) - \rho Tu]\}, \tag{3.34}$$

Where $0 \leq \alpha_n \leq 1$ *is a constant*.
From (3.7), (3.8), (3.6), (3.5), (3.33), (3.34) and using the Lipschitz continuity of the projection P_{Kr} with constant δ, we have

$$\begin{aligned}
||u_{n+1} - u|| &= ||(1-\alpha_n)(u_n - u) + \alpha_n\{P_{Kr}[g(u_n) - \rho Tu_n] - P_{Kr}[g(u) - \rho Tu]\}|| \\
&\quad +\alpha_n||u_n - u - (g(u_n) - g(u)|| \\
&\leq (1-\alpha_n)||u_n - u|| \\
&\quad +\alpha_n||P_{Kr}[g(u_n) - \rho Tu_n] - P_{Kr}[g(u) - \rho Tu]|| + \alpha_n k||u_n - u|| \\
&\leq (1-\alpha_n)||u_n - u|| + \alpha_n\delta||g(u_n) - g(u) + \rho(Tu_n - Tu)|| \\
&\quad +\alpha_n k||u_n - u|| \\
&\leq (1-\alpha_n)||u_n - u|| + \alpha_n k||u_n - u|| \\
&\quad \delta||u_n - u - \big(g(u_n) - g(u)\big)|| + \delta||u_n - u - \rho(T_{un} - Tu)\big)|| \\
&\leq (1-\alpha_n)||u_n - u|| + \alpha_n\{k + \delta[\, k + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}]\}\, ||u_n - u|| \\
&= [1 - \alpha_n(1-\theta)]||u_n - u|| \\
&\leq \prod_{i=0}^{n}[1 - \alpha_i(1-\theta)]\big||u_0 - u|\big|,
\end{aligned}$$

Where, using (3.3), we have

$$\theta = k + \delta\sqrt{1 - 2\rho\alpha + \beta^2\rho^2} < 1$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1 - \theta > 0$, we have $\lim_{n\to\infty}\{\prod_{i=0}^{n}[1 - (1-\theta)\alpha_i]\} = 0$. Consequently the sequence $\{u_n\}$ converges strongly to u. This completes the proof.

Using the technique of the updating the solution, one can rewrite the equation (3.1) in the following form:

$$g(y) = P_{Kr}[g(u) - \rho e^{Tu}]$$

$$g(w) = P_{Kr}[g(y) - \rho e^{Ty}]$$

$$g(u) = P_{Kr}[g(w) - \rho e^{Tw}]$$

which is another fixed point formulation. This fixed-point formulation is used to suggest the following three-step iterative method for solving the nonconvex variational inequality (2.7).

Algorithm 3.12. *For a given $u_n \in H$, find the approximate solution un+1 by the iterative schemes*:

$$y_n = (1-\gamma_n)u_n + \gamma_n\{y_n - g(y_n) + P_{Kr}[g(u_n) - \rho^{T_{un}}]\}$$

$$w_n = (1-\beta_n)u_n + \beta_n\{w_n - g(w_n) + P_{Kr}[g(y_n) - \rho^{T_{yn}}]\}$$

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{u_n - g(u_n) + P_{Kr}[g(w_n) - \rho^{T_{un}}]\}, n = 0,1, \ldots$$

Where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ *are some constants.*

We would like to mention that three-step iterative methods are also known as Noor iteration for solving the variational inequalities and equilibrium problems. One can easily consider the convergence criteria of Algorithm 3.12 using the technique of this paper.

In a similar way, we can suggest the following inertial four step Noor iterations:

Algorithm 3.13. *For given* $u_0, u_1 \in H,$ *find the approximate solution* u_{n+1} *by the iterative schemes:*

$$z_n = (1-\lambda_n)u_n + \lambda_n u_{n-1}$$

$$y_n = (1-\gamma_n)u_n + y_n\{y_n - g(y_n) + P_{Kr}[g(z_n) - \rho^{Tz_n}]\}$$

$$w_n = (1-\beta_n)u_n + \beta_n\{w_n - g(w_n) + P_{Kr}[g(y_n) - \rho^{Ty_n}]\}$$

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{u_n - g(u_n) + P_{Kr}[g(w_n) - \rho^{Tw_n}]\}, n = 0,1, ...,$$

Where $\alpha_n\beta_n\gamma_n \in [0,1]$ *are some constant*

It is worth mentioning that for different and suitable choice of the constantsλ_n $\alpha_n\beta_n$ *and* γ_n one can easily show that the Noor iterations three-step and Noor inertial four step iterations include the Mann and Ishikawa iterations as special cases. Thus we conclude that Noor iterations are more general and unifying ones.

4. Wiener-Hopf Equations Technique

We now consider the problem of solving the nonlinear Wiener-Hopf equations. To be more precise, let $Q_{Kr} = I - P_{Kr}$, where P_{Kr} is the projection operator, and I is the identity operator. For given nonlinear operators T, g, consider the problem of finding $z \in H$ such that

$$E = e^{Tg-1\,P_{Kr^2}} + \rho^{-1}Q_{Kr^z} \tag{4.1}$$

where g^{-1} is the inverse of the operator g. Equations of the type (4.1) are called the exponentially general nonconvex Wiener-Hopf equations. Note that, if r = ∞, then the Wiener-Hopf (4.1) equations are exactly the same Wiener-Hopf equations associated with the general variational inequalities (2.7). For g ≡ I, the identity operator and r = ∞, one can obtain the original Wiener-Hopf equations which were introduced and studied by Shi [39] in conjunction with the variational inequalities. This shows that the original Wiener-Hopf equations are the special case of the exponentially general nonconvex Wiener-Hopf equations (4.1). The Wiener-Hopf equations technique has been used to study and develop several iterative methods for solving variational inequalities and related optimization problems.

We first establish the equivalence between the EGNVI (2.7) and the Wiener-Hopf (4.1) using essentially the projection method. This equivalence is used to suggest and analyze some iterative methods for solving the problem (2.7).

Lemma 4.1. *The nonconvex variational inequality (2.7) has a solution u ∈H : g(u) ∈*K_r*, if and only if, the Wiener-Hopf equations (4.1) have a solution z ∈H provided*

$$g(u) = P_{Kr^z} \tag{4.2}$$

$$z = g(u) - \rho e^{Tu}, \tag{4.3}$$

where ρ > 0 is a constant

Proof. Let u ∈H : g(u) ∈Kr be a solution of (2.7). Then, from Lemma 3.1, we have

$$g(u) = P_{K_r}[g(u) - \rho e^{Tu}]. \tag{4.4}$$

Taking z = $g(u) - \rho e^{Tu}$ in (4.4), we have

$$g(u) = \rho_{K_r z} = \rho_{K_r}[g(u) - \rho e^{Tu}]. \quad (4.5)$$

From (4.4) and (4.5), we have

$$z = g(u) - \rho e^{Tu} = P_{K_r z} - \rho e^{Tg^{-1}P_{K_r z}},$$

Which shows that $z \in H$ is a solution of the Wiener-Hopf equations (4.1). This completes the proof.

From Lemma 4.1, we conclude that the exponentially general nonconvex variational inequality (2.7) and the Wiener-Hopf equations (4.1) are equivalent. This alternative formulation plays an important and crucial part in suggesting and analyzing various iterative methods for solving exponentially general nonconvex variational inequalities and related optimization problems. By suitable and appropriate rearrangement, we can suggest a number of new iterative methods for solving the exponentially general nonconvex variational inequalities.

(I). The Wiener-Hopf equations (4.1) can be written as

$$P_{K_r z} = -\rho e^{TgP_{K_r z}}$$

which implies that, using(4.3)

$$z = P_{K_r z} - \rho e^{Tg^{-1}P_{K_r z}} = g(u) - \rho e^{Tu}.$$

This fixed point formulation enables us to suggest the following iterative method for solving the exponentially general nonconvex variational inequality (2.7).

Algorithm 4.1. *For a given $z_0 \in H$, compute z_{n+1} by the iterative schemes*

$$g = (u_n) = P_{K_r z} \quad (4.6)$$

$$z_{n+1} = (1 - \alpha_n)z_n + \alpha_n\{g(u_n) - \rho e^{Tu_n},\}\ n = 0,1,2, \dots .. \quad (4.7)$$

where $0 \le \alpha_n \le 1$, for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

(II). The Wiener-Hopf equations (4.1) may be written as

$$z = P_{K_r z} - \rho e^{Tg^{-1}P_{K_r z}} + (1 - \rho^{-1})Q_{K_r z}$$

$$= g(u) - \rho e^{Tu} + (1 - \rho^{-1})Q_{K_r z}$$

Using this fixed point formulation, we suggest the following iterative method.

Algorithm 4.2. *For α given $z_0 \in H$, compute z_{n+1} by the iterative schemes*

$$g(u_n) = P_{K_r z}$$

$$z_{n+1} = 1(-\alpha_n)z_n + \alpha_n\left\{g(u_n) - \rho e^{Tu_n} + (1 - \rho^{-1})Q_{K_r z}\right\}\ n = 0,1,2, \dots,$$

where $0 \le \alpha_n \le 1$, for all $n \ge 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

We would like to point out that one can obtain a number of iterative methods for solving the exponentially general nonconvex variational inequality (2.7) for suitable and appropriate choices of the operators T, g and the space H. This shows that iterative methods suggested in this paper are more general and unifying ones. We now study the convergence analysis of Algorithm 4.1. In a similar way, one can analyze the convergence analysis of other iterative methods.

Theorem 4.1. *Let the operator T satisfy all the assumptions of Theorem 3.2. If the condition (3.3) holds,* $\alpha_n \in [0, 1]$, $\forall n \geq 0$, *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$, *then the approximate solution {zn} obtained from Algorithm 4.1 converges to a solution z ∈ H satisfying the Wiener-Hopf equation (4.1) strongly.*

Proof. Let u ∈ H be a solution of (2.7). Then, using Lemma 4.2, we have

$$z = (1 - \alpha_n)z + \alpha_n\{g(u) - \rho e^{Tu}\}, \tag{4.8}$$

where $0 \leq \alpha_n \leq 1, and \ \sum_{n=0}^{\infty} \alpha_n = \infty$.

From (4.8), (4.7), (3.6) and (3.7), we have

$$\begin{aligned} ||z_{n+1} - z|| &\leq (1 - \alpha_n)||z_n - z|| + \alpha_n||g(u_n) - g(u) - \rho(e^{Tu})|| \\ &\leq (1 - \alpha_n)|||z_n - z|| \\ &\quad +\alpha_n\{||u_n - u - \big(g(u_n) - g(u)\big)|| + ||u_n - u - \rho(e^{Tu_n} - e^{Tu})||\} \\ &\leq (1 - \alpha_n)||z_n - z|| + \alpha_n\{k + \sqrt{1 - 2\rho\alpha + \beta^2\rho^2}\}\,||u_n - u||. \end{aligned} \tag{4.9}$$

Also from (4.6), (4.2) and the Lipschitz continuity of the projection operator P_{K_r} with constant δ, we have

$$\begin{aligned} ||u_n - u|| &= ||u_n - u - \big(g(u_n) - g(u)\big)|| + ||P_{K_r}z_n - P_{K_r}z|| \\ &= k||u_n - u|| + \delta||z_n - z|| \end{aligned}$$

From which, we have

$$\|u_n - u\| \leq \frac{\delta}{1-k}\,\|z_n - z\|. \tag{4.10}$$

Combining (4.10), and (4.9), we have

$$\begin{aligned} ||z_{n+1} - z|| &\leq (1 - \alpha_n)||z_n - z|| + \alpha_{n\delta}\frac{k+\sqrt{1-2\rho\alpha+\rho^2\beta^2}}{1-k} \\ &= (1 - -\alpha_n)||z_n - z|| + \alpha_{n\theta_1}||z_n - z||, \end{aligned} \tag{4.11}$$

Where

$$\theta_1 = \delta\frac{k+\sqrt{1-2\rho\alpha+\rho^2\beta^2}}{1-k}$$

From (3.3) and (3.4), we see that $\theta_1 < 1$ and consequently

$$\begin{aligned} ||z_{n+1} - z|| &\leq (1 - \alpha_n)||z_n - z|| + \alpha_n\theta_1||z_n - z|| \\ &= [1 - (1 - \theta_1)\alpha_n]||z_n - z|| + \alpha_n\theta_1]||z_n - z|| \\ &\leq \prod_{i=0}^{n}[1 - (1 - \theta_1)\alpha_i]||z_0 - z||. \end{aligned}$$

Since $\sum_{n=0}^{\infty} \alpha_n$diverges and $1 - 0_1 > 0$, we have limn→∞ $\prod_{i=0}^{n}[1 - (1 - \theta_1)\alpha_i] = 0$.

Consequently the sequence {zn} converges strongly to z in H, the required result.

5. Dynamical Systems

In this section, we consider the projected dynamical system associated with the general variational inequalities. Using the fixed-point formulation of the variational inequalities, Dupuis et al. [5] introduced and considered the projected dynamical systems, which the right hand side of the ordinary differential equation is a projected operator associated with variational inequalities. The innovative and novel feature of a projected dynamical system is that its set of stationary points corresponds to the set of solutions of the corresponding variational inequality problem. Hence, equilibrium and nonlinear problems arising in various branches in pure and applied sciences, which can be formulated in the setting of the variational inequalities, can now be studied in the more general setting of dynamical systems. It have been shown [10], [5], [29], [35] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. In recent years, much attention has been given to study the globally asymptotic stability of these projected dynamical systems. In Section 3, we have shown that the exponentially general nonconvex variational inequalities are equivalent to the fixed-point. We use this equivalent formulation to suggest and analyze the projected dynamical system associated with the exponentially general nonconvex variational inequalities (2.7).

$$\frac{du}{dt} = \lambda\{P_{K_r}[g(u) - pe^{Tu}] - g(u)\},\ u(t_0) = u_0 \in H, \tag{5.1}$$

where λ is a parameter. The system of type (5.1) is called the exponentially projected general dynamical system. Here the right hand side is related to the projection operator and is discontinuous on the boundary. It is clear from the definition that the solution to (5.1) always stays in the constraint set. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution on the given data can be studied.

The equilibrium points of the dynamical system (5.1) are naturally defined as follows.

Definition 5.1. *An element $u \in H, g(u) \in K_r$ is an equilibrium point of the dynamical system (5.1), if $\frac{du}{dt} = 0,$ that is,*

$$P_{K_r}[g(u) - \rho e^{Tu}] - g(u) = 0,$$

Thus it is clear that $u \in H, g(u) \in K_r$ is a solution of the general variational inequality (2.7), if and only if, $u \in H, g(u) \in K_r$ is an equilibrium point. In a similar way, one can define the concept of equilibrium points for other dynamical systems.

Definition 5.2 The dynamical system is said to converge to the solution set S^{**} of (2.7), if , irrespective of the initial point, the trajectory of the dynamical system satisfies.

$$\lim_{t\to\infty} dist(u(t), S^{**}) = 0, \tag{5.2}$$

Where

$$dist(u, S^*) = inf_{v\in S^*}||u - v||.$$

It is easy to see, if the set S^* has a unique point u^* , then (5.2) implies that

$$\lim_{t\to\infty} dist(u(t) = u^*.)$$

If the dynamical system is still stable at u^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at u^*.

Definition 5.3. The dynamical system is said to be globally exponentially stable with degree η at u^* , if, irrespective of the initial point, the trajectory of the system satisfies.

$$||u(t) - u^*|| \le \mu_1 ||u(t_0) - u^*|| \exp(-\eta(t - t_0)),\ \forall t \ge t_0,$$

Where μ_1 and η are positive constants independent of the initial point.

Lemma 5.1. *(Gronwall Lemma)* [54]. *Let $\hat{u}$ and $\hat{v}$ be real-valued nonnegative continuous functions with domain $\{t: t \le t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If for $t \ge t_0$,*

$$\hat{u} \leq \alpha(t) + \int_{t_0}^{t} \hat{u}(s)\hat{v}(s)ds,$$

then

$$\hat{v} \leq \alpha(t)\exp\{\int_{t_0}^{t} \hat{v}(s)ds\},$$

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast. We now show that the trajectory of the solution of the general dynamical system (5.1) converges to the unique solution of the general variational inequality (2.7). The analysis is in the spirit of Noor [?,?,?,?] and Xia and Wang [?]. In a similar way, one can consider the other dynamical systems.

Theorem 5.1 *Let the operators T, g : H −→ H be both Lipschitz continuous with constants β > 0 and μ > 0 respectively. Then, for each* $u_{(0)} \in H$*, there exists a unique continuous solution u(t) of the dynamical system (5.1) with* $u_{(0)} = u_{(0)}$ *over* $t_{(0)}, \infty)$*.*

Proof. Let

$$G(u) = \lambda\{P_{K_r}[g(u) - \rho e^{Tu}] - g(u)\}.$$

Where $\lambda > 0$ is a constant. For all $u,v \in H$, we have

$$\begin{aligned} ||G(u) - G(v)|| &\leq \lambda\{||P_{K_r}[g(u) - \rho e^{Tu}] - P_{K_r}[g(v) - \rho e^{Tu}]|| + ||g(u) - g(v)||\} \\ &\leq \delta\lambda||g(u) - g(v)|| + \delta\lambda\rho||e^{Tu} - e^{Tu}|| + \lambda||g(u) - g(v)|| \\ &\leq \lambda\{\mu\delta + \mu + \beta\rho\}||u - v||. \end{aligned}$$

This implies that the operator *G(u)* is a Lipschitz continuous in *H*, and for each *u0* $\in H$, there exists a unique and continuous solution *u(t)* of the dynamical system (5.1), defined on an interval $t_0 \leq t < T_1$ with the initial condition $u(t_0) = u_0$ *Let* $[t_0, T_1$ be its maximal interval of existence. Then we have to show that T1 = ∞. Consider , for any $u \in H$,

$$\begin{aligned} ||G(u)|| &= \lambda||P_{K_r}[g(u) - \rho T_u] - g(u)|| \\ &\leq \lambda\{||P_{K_r}[g(u) - \rho T_u] - P_{K_r}[g(u)]|| + ||P_{K_r}[g(u)||\} \\ &\leq \delta\lambda\{||\rho||\text{Tu} - P_{K_r}[g(u)]|| - P_{K_r}[0]|| + ||P_{K_r}[0] - g(u)||\} \\ &\leq \delta\lambda\{(\rho\beta + 2\mu)||u|| + ||P_{K_r}[0]||\} \end{aligned}$$

Then

$$\begin{aligned} ||u(t)|| &\leq ||u_0|| + \int_{t0}^{t} ||Tu(s)||ds \\ &\leq (||u_0|| + K_1(t - t_0)) + K_2\int_{t_0}^{t} ||u(s)||ds, \end{aligned}$$

Where $K_1 = \lambda||P_k[0]||$ *and* $K_2 = \delta\lambda(\rho\beta + 2\mu)$. Hence by the Gronwall lemma 5.1, we have

$$||u(t) \leq \{||u_0|| + K_1(t - t_0)\}e^{K2(t-t_0)}, \ t \in [t_0, T_1).$$

This shows that the solution is bounded on $[t_0, T_1). So\ T_1 = \infty$.

Theorem 5.2. *Let the operators T, g : H → H be both Lipschitz continuous withconstants β > 0 and μ > 0 respectively. If the operator g : H → H is strongly monotone with constant γ > 0 and λ < 0,* *then the dynamical system (5.1) converge globally exponentially to the unique solution of the general variational inequality (2.7).*

Proof. Since the operators *T*, g are both Lipschitz continuo, it follows from Theorem 5.1 that the dynamical system (5.1) has unique solution *u(t)* over $[t_0, T_1)$ for any fixed $u_0 \in H$. Let *u(t)* be a solution of the initial value problem (5.1). For a given $u^* \in$ satisfying (2.7), consider the Lyapunov function.

$$L(u) = \lambda ||u(t) - u^*||^2, \; u(t) \in H. \tag{5.3}$$

From (5.1) and (5.3), we have

$$\begin{aligned} \frac{dL}{dt} &= 2\lambda(u(t) - u^*, P_{K_r}[g(u(t)) - \rho e^{Tu(t)}] - g(u(t))) \\ &= -2\lambda(u(t) - u^*, g(u(t)) - g(u^*)) \\ &\quad +2\lambda(u(t) - u^*, P_{K_r}[g(u(t)) - \rho e^{Tu(t)}] - g(u^*)) \\ &\leq -2\lambda\gamma ||u(t) - u^*||^2 \\ &\quad +2\lambda(u(t) - u^*, P_{K_r}[g(u(t)) - \rho e^{Tu(t)}] - g(u^*)), \end{aligned} \tag{5.4}$$

Where $u^* \in H$ is a solution of (2.7). Thus

$$g(u^*) = P_{K_r}[g(u^*) - \rho T u^*].$$

Using the Lipschitz continuity of the operators *T*, g, we have

$$\begin{aligned} ||P_{K_r}(g(u) - \rho Tu] - P_{K_r}[g(u^*]|| &\leq \delta ||g(u) - g(u^*) - \rho(e^{Tu} - e^{Tu^*})|| \\ &\leq \delta(\mu + \rho\beta)||u - u^*||. \end{aligned} \tag{5.5}$$

From (5.4) and (5.5), we have

$$\frac{d}{dt} ||u(t) - u^*|| \leq 2\alpha\lambda ||u(t) - u^*||,$$

Where

$$\alpha = \delta(\mu + \rho\beta) - \gamma.$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$||u(t) - u^*|| \leq u(t_0) - u^*||e^{-\alpha\lambda 1(t - t_0)}$$

which shows that the trajectory of the solution of the dynamical system (5.1) converges globally exponentially to the unique solution of the exponentially general nonconvex variational inequality (2.7).

We now use the fixed point formulation to suggest and consider a new second order projection dynamical system associated with exponentially general nonconvex variational inequalities (2.7). We use this dynamical system to suggest and investigate some inertial proximal methods for solving the problem (2.7). These inertial implicit methods are constructed using the central finite difference schemes and its variant forms. To be more precise, we consider the problem of finding u ∈ H such that

$$\gamma \frac{d^2 g(u)}{dt^2} + \frac{dg(u)}{dt} + g(u) = P_{K_r}[g(u) - \rho e^{Tu}],$$

$$u(t_0) = \alpha, \hat{u}(t_0) = \beta, \quad (5.6)$$

where γ > 0 and ρ > 0 are constants. Problem (5.6) is called second order dynamical system.

If γ = 0, then dynamical system (5.6) reduces to dynamical system (5.1). We discretize the second-order dynamical systems (5.6) using central finite difference and backward difference schemes to have.

$$\gamma \frac{g(u_{n+1}) - 2g(u_n) + g(u_n) - g(u_{n-1})}{h^2 h} = P_{K_r}[g(u_n) - \rho e^{T(u_{n+1})}], \quad (5.7)$$

Where h is the step size

If $\gamma = 1, h = 1$, then, from equation(5.7) we have

Algorithm 5.1. *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$g(u_{n+1}) = P_{K_r}[g(u_n) - \rho e^{T(u_{n+1})}], \ n = 0,1,2, \ldots$$

which is the extragradient method of Korpelevich [8] for solving the exponentially general nonconvex variational inequalities. Algorithm 5.1 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the two-step inertial method.

Algorithm 5.2. *For given $u_0, u_1 \in H$, compute u_{n+1} by the iterative scheme*

$$y_n = (1 - \theta_n)u_n + \theta_n u_n - 1$$

$$g(u_{n+1}) = P_{K_r}[g(u_n) - \rho e^{T(y_n)}],$$

Where θ_n is a constant. Similarly, we suggest the following iterative method.

Algorithm 5.3. *For given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$u_{n+1} = P_{K_r}[g(u_{n+1}) - \rho e^{T(u_{n+1})}], \ n = 0,1,2, \ldots$$

which is known as the double projection method, introduced and studied by Noor [13] and can be written as

Algorithm 5.4. *For a given u_0, $u_1 \in H$, compute u_{n+1} by the iterative scheme*

$$y_n = (1 - \theta_n)u_n + \theta u_{n-1}$$

$$g(u_{n+1}) = P_{K_r}[g(y_n) - \rho e^{T(y_n)}], \ n = 0,1,2, \ldots$$

Which is called the two-step inertial iterative Noor method.

Problem (5.6)can be rewritten as

$$\gamma \frac{d^2 g(u)}{dt^2} + \frac{dgu)}{dt} + g(u) = P_{K_r}[g((1 - \theta_n)) - \rho e^{T((1-\theta_2)u + \theta_n u)}],$$

$$u_{(t_0)} = \alpha, \hat{u}(t_0) = \beta, \quad (5.8)$$

Where $\gamma > 0, \theta_n$ *and* $\rho > 0$ *are constants.*

Discretising the system (5.8), we have

$$\gamma \frac{g(u_{n+1}) - 2g(u_n) + g(u_{n-1})}{h^2} + \frac{g(u_n) - g(u_{n-1})}{h} + g(u_n)$$

$$= P_{K_r}[g((1 - \theta_n)u_n + \theta_n u_n - 1)) - \rho e^{((1-\theta_n)\mu_n + \theta_n \mu_n - 1)}]$$

From which, for $\gamma = 0, \ h = 1$, we have

Algorithm 5.5. *For a given* $u_0, u_1 \ \in H$, *compute* μ_{n+1} *by the iterative scheme*

$$g(u_{n+1}) = P_{K_r}[g\big((1-\theta_n)u_n + \theta_n u_n - 1\big) - \rho e^{Ty_{n})}]$$

Or equivalenty

Algorithm 5.6. *For a given* $u_0, u_1 \ \in H$, *compute* u_{n+1} *by the iterative scheme*

$$y_n = (1-\theta_n)u_n + \theta_n u_n - 1$$

$$g(u_{n+1}) = P_{K_r}[g(y_n) - \rho e^{Ty_n}$$

which is called the new inertial iterative method for solving the exponentially general nonconvex variational inequality.

We discretize the second-order dynamical systems (5.6) using central finite difference and backward difference schemes to have

$$\gamma \frac{g(u_{n+1}) - 2g(u_n) + g(u_{n-1})}{h^2} + \frac{g(u_n)}{h} + g(u_{n+1}) = P_{K_r}[g(u_n) - \rho e^{T(u_{n+1})}]$$

Where h is the step size

Using this discrete form, we can suggest the following an iterative method for solving the variational inequalities (2.7).

Algorithm 5.7. *For given* $u_0, u_1 \ \in H$, *compute* u_{n+1} *by the iterative scheme*

$$g(u_{n+1})$$

$$= P_{K_r}[g(u_n) - \rho e^{T(u_{n+1})} - \frac{\gamma g(u_{n+1}) - (2\gamma - h)g(u_n) + (\gamma - h)g(u_{n-1})}{h^2}]$$

Algorithm 5.7 is called the inertial proximal method for solving the exponentially general nonconvex variational inequalities and related optimization problems. This is a new proposed method.

We can rewrite the Algorithm 5.7 in the equivalent form as follows:

Algorithm 5.8. *For a given* $u_0 \ \in H$, *compute* u_{n+1} *by the iterative scheme*

$$(\rho e^{Tu_{n+1}} + g(u_{n+1}) - \frac{\gamma g(u_{n+1}) - (2\gamma - h)g(u_n) + (\gamma - h)g(u_{n-1})}{h^2}, v - g(u_{n+1}))$$

$$\geq 0, \forall v \in K_r$$

We note that, for $\gamma = 0$, Algorithm 5.8 reduces to the following iterative method for solving the exponentially general nonconvex variational inequalities (2.7).

Algorithm 5.9. *For given* $u_0, u_1 \ \in H$, *compute* u_{n+1} *by the iterative scheme*

$$g(u_{n+1}) = P_{K_r}\left[g(u_n) - \rho e^{Tu_{n+1}} - \frac{g(u_n) - g(u_{n-1})}{h}\right], \ n = 0,1,2,\ldots$$

We again discretize the second-order dynamical systems (5.6) using central difference scheme and forward difference scheme to suggest the following inertial proximal method for solving (2.7).

Algorithm 5.10. *For a given* $u_0 \ \in H$, *compute* u_{n+1} *by the iterative scheme*

$$g(u_{n+1}) = P_{K_r}\left[g(u_{n+1}) - \rho e^{T(u_{n+1})} - \frac{(\gamma + h)g(u_{n+1}) - (2\gamma + h)g(u_n) + \gamma g(u_{n-1})}{h^2}\right].$$

If $\gamma = 0$, then Algorithm 5.10 collapses to:
Algorithm 5.11. *For a given* $u_0 \in H$, *compute* u_{n+1}*by the iterative scheme*

$$g(u_{n+1}) = P_{K_r}\left[g(u_{n+1}) - \rho e^{T(u_{n+1})} - \frac{g(u_{n+1}) - g(u_n)}{h}\right], \ n = 0,1,2,\ldots$$

Algorithm 5.10 is an proximal method for solving the nonconvex variational inequalities. Such type of proximal methods were suggested by Noor [48] using the fixed point problems. In brief, by suitable descritization of the second-order dynamical systems (5.6), one can construct a wide class of explicit and implicit method for solving inequalities. We now consider the convergence criteria of the Algorithm 5.8 using the technique of Alvarez [50], Noor [21] and Noor et al. [35].

Theorem 5.3. *Let $u \in H$ be the solution of quasi variational inequality (2.7). Let un+1 be the approximate solution obtained from (5.9). If T is monotone, then*

$$(h + h^2)||g(u) - g(u_{n+1})||^2 \leq (\gamma + h^2)||g(u) - g(u_n)||^2 - (\gamma + h^2)||g(u_{n+1}) - g(u_n)||^2$$
$$+(\gamma - h)||g(u_{n+1}) - g(u_n)||^2. \qquad (5.9)$$

Proof. Let $u \in H$ be the solution of exponentially general nonconvex variational inequality (2.7). Then

$$\left(\rho e^{Tu}, g(v) - g(u)\right) \geq 0, \ \forall v \in K_r, \qquad (5.10)$$

since T is a exponentially general monotone operator.
Setting $v = u_{n+1}$ in (5.10), we have

$$(\rho e^{T(u_{n+1})}, g(u_{n+1}) - g(u)) \geq 0. \qquad (5.11)$$

Taking v = u in (5.9), we have

$$(\rho e^{T(u_{n+1})} + \frac{(\gamma + h^2)g(u_{n+1}) - (2\gamma - h + h^2)g(u_n) + (\gamma - h)g(u_{n+1})}{h^2}, g(u) - g(u_{n+1}))$$
$$\geq 0. \qquad (5.12)$$

From (5.11) and (5.12), we obtain

$$((\gamma + h^2)g(u_{n+1}) - (2\gamma - h + h^2)g(u_n) + (\gamma - h)g(u_{n+1}), g(u) - g(u_{n+1})) \geq 0.$$

Thus

$$0 \leq (\gamma + h^2)(g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1}))$$
$$+(\gamma - h)(g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1}))$$
$$\leq (\gamma + h^2)||g(u) - g(u_n)||^2 - (\gamma + h^2)||g(u_{n+1}) - g(u_n)||^2$$
$$-(\gamma + h^2)||g(u) - g(u_{n+1})||^2 + (\gamma - h)||g(u_{n+1}) - g(u_n)||^2$$
$$+(\gamma - h)||g(u) - g(u_{n+1})||^2$$
$$= (\gamma + h^2)||g(u) - g(u_n)||^2 - (\gamma + h^2)||g(u_{n+1}) - g(u_n)||^2$$
$$+(\gamma - h)||g(u_{n+1}) - g(u_n)||^2 - h(1 + h)||g(u) - g(u_{n+1})||^2, \qquad (5.13)$$

Where we have used the following inequalities

$$2(u, v) = ||u + v||^2 - ||v||^2 - ||v||^2, \ \forall v, u \in H$$

and

From (5.13), we have

$$(h+h^2)||g(u)-g(u_{n+1})||^2 \le (\gamma+h^2)||g(u)-g(u_n)||^2-(\gamma+h^2)||g(u_{n+1})-g(u_n)||^2$$

$$+(\gamma-h)||g(u_{n+1})-g(u_n)||^2$$

the required (5.9).

Theorem 5.4. *Let $u \in K_r$ be a solution of exponentially general nonconvex variational inequality (2.7). Let μ_{n+1} be the approximate solution obtained from (5.9). If the operator T is monotone and g^{-1}exists, then un+1 converges to $u \in K_r$ satisfying (2.7).*

Proof. Let $u \in K_r$ be a solution of (2.7). From (5.10), it follows that the sequence {||u-u_i||} is non-increasing and consequently,$\{u_n\}$ is bounded. Also from(5.10), we have.

$$\sum_{i=1}^{\infty} ||g(u)_n-g(u_{n+1})||^2 \le ||g(u)-g(u_1)||^2+\frac{\gamma-h}{\gamma+h^2}||g(u_0)-g(u_1)||^2$$

Which implies that

$$\lim_{n\to\infty} ||u_{n+1}-u_n||^2=0, \tag{5.14}$$

where we have used the fact that g^{-1} exist. Since sequence $\{u_i\}_{i=1}^{\infty}$ is bounded, so there exists a cluster point $\hat{u}$ to which the subsequence $\{u_{ik}\}_{k=i}^{\infty}$ converges. Replacing u_n by u_{ni} in (3.2), taking the limit as $n_j \to \infty$ and using (5.14),we have

$$\left(e^{T(\widehat{u})}, g(v)-g(\hat{u})\right) \ge 0, \forall v \in K_r,$$

Which implies that $\hat{u}$ solves (2.7) and

$$||g(u_{n+1})-g(u)||^2 \le \frac{\gamma+h^2}{h+h^2}||g(u)-g(u_n)||^2+\frac{\gamma-h}{h+h^2}||g(u_n)-g(u_{n+1})||^2$$

$$\le ||g(u)-g(u_n)||^2$$

Using this inequality, one can show that the cluster point $\hat{\mu}$ is unique and

$$\lim_{n\to\infty} u_{n+1}=\hat{u}$$

In this section, we have applied the dynamical systems associated with exponentially general nonconvex variational inequalities to suggest some iterative schemes for solving the variational inequalities. We have only considered the theoretical aspects of the proposed iterative methods. It is an interesting open problem to consider the implementable of these numerical methods. Comparison with other methods need further research efforts.

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