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Adaptive PID Fault-Tolerant Tracking Controller for Takagi-Sugeno Fuzzy Systems with Actuator Faults: Application to Single-Link Flexible Joint Robot

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ABSTRACT

This paper considers the problem of Fault Tolerant Tracking Control (FTTC) strategy design for nonlinear systems using Takagi-Sugeno (T-S) fuzzy models with measurable premise variables affected by actuator faults subject to unknown bounded disturbances (UBD). Firstly, the Adaptive Fuzzy Observer (AFO) is proposed to estimate the faults. Based on the information provided by this observer, an active fault tolerant tracking controller described by an adaptive Proportional-Integral-Derivative (PID) structure has been developed to compensate for the actuator fault effects and to guarantee the trajectory tracking of desired outputs to the reference model despite the presence of actuator faults. The stability and the trajectory tracking performances of the proposed approach are analyzed based on the Lyapunov theory. Sufficient conditions can be obtained and solved for the design of the controller, and the observer gains using Linear Matrix Inequalities (LMIs). Finally, the effectiveness of the proposed technique is illustrated by using a single-link flexible joint robot.

1. Introduction

In recent decades, with the challenge of increasing demand for reliability and safety of various industrial control processes, Fault Diagnosis (FD) and Fault Tolerant Control (FTC) has attracted more and more attention. More precisely, the FTC system is a control system that can accommodate system component faults and is able to maintain system stability and performance despite the presence of faults [1]–[4]. Indeed, real-world physical systems are generally nonlinear in nature. Moreover, it is well known that it is difficult to directly extend the existing FTC methods of linear systems to nonlinear ones.

Furthermore, a large class of nonlinear systems can be described by using suitable Takagi-Sugeno (T-S) fuzzy models [5]. The main idea of the T-S fuzzy modeling method is to decompose the model of a nonlinear system into several locally linearized models involving nonlinear weighting functions.

Accordingly, several researchers have been interested in developing new FTC approaches based on T-S fuzzy representation. For example, in [6], a fuzzy-reduced-order robust state and fault estimation observer has been proposed to estimate the system states, sensor, and actuator faults. In addition, an observer-based fault-tolerant controller has been designed to guarantee the robustly asymptotically stability of the closed-loop system. Also, authors in [7] developed a method for designing a robust fault-tolerant tracking control technique for nonlinear uncertain systems approximated by Takagi–Sugeno fuzzy models with unmeasurable premise variables subject to sensor faults. The authors in [8] have proposed a fault-tolerant tracking control for uncertain T-S fuzzy systems affected simultaneously by sensor faults, actuator faults, and unknown bounded disturbance. A robust descriptor adaptive observer-based FTTC has been proposed to estimate unmeasured system states and both sensor and actuator fault vectors simultaneously, which have then been used in order to compensate for the effects of the faults and to track the reference model states despite the presence of external disturbances and uncertainties. A robust adaptive observer-based fault tolerant control is presented in [9] for nonlinear systems described by a T-S fuzzy model affected by both sensor and actuator faults in the presence of external disturbances. So, a robust adaptive observer has been developed to simultaneously estimate the state, sensor, and actuator faults vectors. Based on this observer, the fuzzy control law can preserve the stability and the performance of the closed-loop system and compensate for fault effects. The authors [10] have presented a combined state, faults estimation, and fault tolerant control method based on Proportional Multi-Integral Observer for fuzzy descriptor systems affected by actuator faults.

Today, the standard Proportional-Integral-Derivative (PID) control schemes continue to provide the simplest yet most effective solutions to most control engineering applications. The popularity of the PID controllers is mainly due to their simplicity and easy adaptation to control a large number of systems. Recently, the PID controller has been used for FTC in order to compensate for the fault effects and stabilize the closed-loop system. Indeed, authors in [11] have designed an adaptive PID actuator fault tolerant control based on the sliding mode approach for a single-link flexible manipulator. In [12], an adaptive PID-FTTC for DC series Motor represented by T-S model subject to actuator faults has been proposed using the adaptive fuzzy observer. Moreover, in [13], the authors consider the problem of model reference tracking based on the design of an Active Fault Tolerant Control for polytopic Linear Parameter Varying (LPV) systems affected by actuator faults and unknown inputs. The authors in [14] have designed a descriptor observer-based sensor and actuator fault-tolerant tracking control for LPV systems. The descriptor observer is able de estimates sensor and actuator faults affecting systems simultaneously, and a proposed PID-Active FTTC law which allows for tolerance of actuator faults and ensures the stability and trajectory tracking performances, especially in terms of accuracy and rapidity.

Until now, there are very few works dealing with the tracking FTC design with PID controllers for T-S fuzzy models in the realm of control, and it is more difficult and more general than the stabilizing FTC. This field remains very important, which motivates our study in this work. Therefore, the main contribution of this paper consists in designing a PID-FTTC closed loop system scheme based on the synthesis of an Adaptive Fuzzy Observer (AFO), which estimates states and actuator faults. The principle of the proposed method is to use a T-S fuzzy model to represent the nonlinear dynamics of the system. Based on the estimated actuator faults, the designed controller is automatically reconfigured in order to compensate for the fault effects, keep the system stable, and to ensure tracking trajectories with good accuracy performances. However, in this work, we compute both controller and observer gains by solving a set of LMIs in an integrated and in one step. Moreover, a single-link flexible joint robot system with disturbance and actuator faults is used to validate the effectiveness of the developed PID-FTTC scheme.

The remaining of this paper is structured in the following order. Section 2 provides the system description and the problem statement. Section 3 is dedicated to the design of an observer-based PID-FTTC to stabilize the closed loop system in the presence of actuator faults. In section 4, simulation results are given, using the single-link flexible joint robot example in order to illustrate the importance and the effectiveness of the proposed PID-FTTC scheme with actuator faults. To close the paper, some conclusions are given in section 5.

Notations: In this paper, we note I as an identity matrix with an appropriate dimension. In a matrix, * denotes the transposed element in the symmetric position.

2. Preliminaries and Problem Formulation

2.1. T-S System Modeling

Generally, most physical systems are inherently nonlinear. To overcome these non-linearities, they can be described by fuzzy models based on the T-S approach. The T-S fuzzy model is a set of linear time-invariant systems connecting via membership functions. Actually, three techniques have been used to build a T-S fuzzy model from existing nonlinear models. An interesting method is well-known in the nonlinear sector transformation [15]. These transformations allow obtaining an exact T-S representation of a nonlinear system model without information loss on a compact set of the state space.

Consider the following nonlinear system:

$$\begin{cases} \dot{x}(t) = f(x(t)) + g(x(t))u(t) \\ \quad\quad y(t) = Cx(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the input vector, $y(t) \in \mathbb{R}^m$ represents the output measurement vectors and $C \in \mathbb{R}^{m\times n}$ is the output matrix. In addition, $f(.)$and $g(.)$ represent the nonlinear functions.

The T-S fuzzy model uses a set of fuzzy if-then rules, which represent local linear input-output relations of a nonlinear system. The i^{th} rule of the T-S model is given as follows:

Rule i:

If $\xi_1(t)$ is $F_1^i(\xi_1(t))$ **and** $\xi_2(t)$ is $F_2^i(\xi_2(t))$….**and** $\xi_p(t)$ is $F_p^i\big(\xi_p(t)\big)$ **THEN**

$$\begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t) \\ y(t) = C_i x(t) \end{cases} \tag{2}$$

where F_p^i are the membership functions of fuzzy sets, $i \in \{1,2,\ldots..,h\}$, r is the number of rules, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{p\times n}$ and $\xi_1(t), \xi_2(t), \ldots\ldots, \xi_p(t)$ are the premise variables which can be dependent on the inputs, the outputs, or the states. The global T-S structure is given by:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{h} \mu_i(\xi(t))\,(A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{cases} \tag{3}$$

where $\mu_i(\xi(t))$ are the weighting functions indicate the activation degree of the i^{th} associated local model. These nonlinear functions verify all time the convex sum propriety:

$$\begin{cases} \sum_{i=1}^{h} \mu_i(\xi(t)) = 1 \\ 0 \le \mu_i(\xi(t)) \le 1 \end{cases} \tag{4}$$

2.2. Problem Formulation

Our goal in this work is to synthesize an FTC law based on an adaptive PID controller. The controller design ensures automatic reconfiguration to the nominal control input after the apparition of actuator faults to guarantee the asymptotic stability in a closed loop and to minimize the tracking trajectory error between the reference model outputs and faulty system outputs. The bloc scheme of the proposed PID-FTTC closed loop system is given in Fig. 1. The elements of the FTTC loop include the Adaptive Fuzzy Observer, the adaptive PID controller, and the model reference. All these elements are described using the T-S fuzzy technique. Firstly, the AFO is developed in order to estimate states and actuator faults for T-S fuzzy systems with external disturbances. Then, a PID-FTTC is designed to compensate for the actuator faults effects, stabilize the closed-loop system, and ensure very good tracking performances.

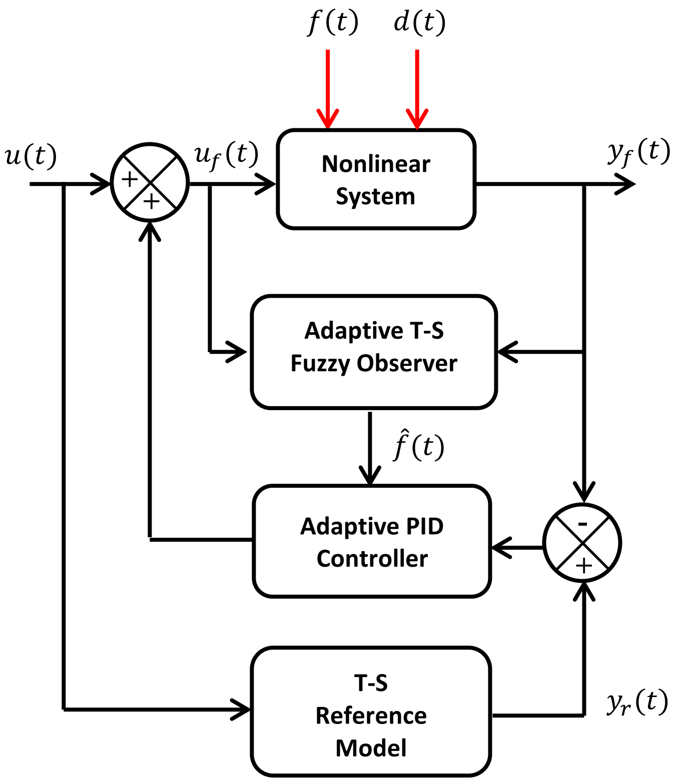

Fig. 1. PID-FTTC closed loop system scheme

The model reference is considered given by the following T-S fuzzy healthy model defined as:

$$\begin{cases} \dot{x}_r(t) = \sum_{i=1}^{h} \mu_i(\xi(t))\,(A_i x_r(t) + B_i u(t)) \\ y_r(t) = C x_r(t) \end{cases} \tag{5}$$

Where $x_r(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^p$ and $y_r(t) \in \mathbb{R}^m$ represent respectively state vector, the bounded control input vector, and the measured output vector of the reference model.

In the case of actuator faults and unknown bounded disturbances, the faulty nonlinear system described by the T-S fuzzy model can be expressed as follows:

$$\begin{cases} \dot{x}_f(t) = \sum_{i=1}^{h} \mu_i(\xi(t))(A_i x_f(t) + B_i(u_f(t) + f(t)) + R_i d(t)) \\ y_f(t) = C x_f(t) \end{cases} \tag{6}$$

where $x_f(t) \in \mathbb{R}^n$ represents the faulty state vector, $u_f(t) \in \mathbb{R}^p$ is the FTC law vector to be designed, $f(t) \in \mathbb{R}^p$ is the actuator fault vector, $d(t) \in \mathbb{R}^q$ represents the unknown bounded disturbance vector and $y_f(t) \in \mathbb{R}^m$ stands the measured output vector.

Firstly, an Adaptive Fuzzy Observer is developed in order to estimate faulty states and actuator faults for T-S fuzzy systems with external disturbances. The fault estimation $\hat{f}(t)$ provided by an AFO that is given as follows:

$$\begin{cases} \dot{z}(t) = \sum_{i=1}^{h} \mu_i(\xi(t)) \left(N_i z(t) + G_i u_f(t) + L_i y_f(t) + B_i \hat{f}(t)\right) \\ \hat{x}_f(t) = z(t) + T_2 y_f(t) \\ \hat{y}_f(t) = C\hat{x}_f(t) \\ \dot{\hat{f}}(t) = \Gamma \sum_{i=1}^{h} \mu_i(\xi(t)) \left(\Phi_i(\dot{e}_y(t) + \sigma e_y(t))\right) \\ e_y(t) = y_f(t) - \hat{y}_f(t) \end{cases} \tag{7}$$

where $z(t) \in \mathbb{R}^n$ is the observer state vector, $\hat{x}(t) \in \mathbb{R}^n$ the estimated state vector, $\hat{y}(t) \in \mathbb{R}^m$is the estimated output vector, $\hat{f}(t) \in \mathbb{R}^q$is the estimated actuator faults and $N_i \in \mathbb{R}^{n\times n}$, $G_i \in \mathbb{R}^{n\times p}$, $L_i \in \mathbb{R}^{n\times m}$, $\Phi_i \in \mathbb{R}^{q\times m}$ and $T_2 \in \mathbb{R}^{n\times m}$ are the observer gain matrices to be determined. The matrix $\Gamma \in \mathbb{R}^{p\times p}$ is a symmetric positive definite learning rate matrix and σ is a positive scalar.

Now, based on the presented strategy in [9], the designed PID-FTTC control law has a structure given by:

$$u_f(t) = \sum_{i=1}^{h} \mu_i(\xi(t)) \left(K_{Pi} e_{yr}(t) + K_{Ii} \int_0^t e_{yr}(t)\, dt + K_{Di} \frac{de_{yr}(t)}{dt} - K_{fi}\hat{f}(t)\right) + u(t) \tag{8}$$

where $K_{Pi} \in \mathbb{R}^{p\times m}$, $K_{Ii} \in \mathbb{R}^{p\times m}$, $K_{Di} \in \mathbb{R}^{p\times m}$ and $K_{fi} \in \mathbb{R}^{p\times q}$ are the gain matrices to be determined.

Adaptive Fuzzy Observer (7) based fault tolerant tracking control (8) has been designed for nonlinear systems represented by T-S fuzzy models affected by actuator faults and external disturbances. First of all, the AFO can estimate system states and actuator faults accurately. After that, the fuzzy controller can compensate for fault effects, guarantee the stabilization of the closed-loop fuzzy system and ensure the tracking of the desired reference trajectories with good accuracy and rapid responses.

Furthermore, to develop the PID-FTTC closed loop system, the following assumptions are necessary:

Assumption 1 [16]. We assume that the actuator fault distribution matrices B_i are of full column rank $\forall i = 1,\dots,h$:

$$rank(CB_i) = rank(B_i) = p \tag{9}$$

Assumption 2 [16]. The pair (A_i, C) is observable $\forall i = 1,\dots,h$.

$$rank \begin{bmatrix} C \\ CA_i \\ \vdots \\ CA_i^{n-1} \end{bmatrix} = n \tag{10}$$

Assumption 3 [16]. The input vector $u(t)$ is bounded as $\|u(t)\| \le \alpha_1$. The fault $f(t)$ satisfies $\|f(t)\| \le \alpha_2$, its derivative is bounded such that $\|\dot{f}(t)\| \le \alpha_3$and the unknown input vector $d(t)$ verifies $\|d(t)\| \le \alpha_4$, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \ge 0$.

3. Design and Analysis of PID-FTTC for T-S Fuzzy System

3.1. System Reconfiguration

To analyze the reconfigured system, let's consider the following augmented model $\tilde{x}(t)$ which includes the faulty state vector $x_f(t)$, the state of the tracking error $e_t(t)$, the state of the estimation error $e_s(t)$ and the fault estimation error $e_f(t)$ as follows:

$$\tilde{x}(t) = \begin{pmatrix} x_f(t) \\ e_t(t) \\ x_t(t) \\ e_s(t) \\ e_f(t) \end{pmatrix} = \begin{pmatrix} x_f(t) \\ x_r(t) - x_f(t) \\ \int_0^t e_t(t) \\ x_f(t) - \hat{x}_f(t) \\ f(t) - \hat{f}(t) \end{pmatrix} \tag{11}$$

Now, a new state vector $x_t(t)$ is used such that its dynamic is given by [17]

$$\dot{x}_t(t) = e_t(t) \tag{12}$$

By using the state space representations (5) and (6) and the expression of $e_t(t)$ in (11), the error $e_{yr}(t)$ can be written as:

$$e_{yr}(t) = y_r(t) - y_f(t) = Ce_t(t) \tag{13}$$

Knowing the expressions of $e_t(t)$, $\dot{x}_t(t)$ and using (11), the FTC law (8) can be written as:

$$u_f(t) = \sum_{i=1}^{h} \mu_i(\xi(t)) \left(K_{Pi}Ce_t(t) + K_{Ii}Cx_t(t) + K_{Di}C\dot{e}_t(t) + K_{fi}e_f(t) - K_{fi}f(t)\right) + u(t) \tag{14}$$

The dynamic of the tracking error is:

$$\dot{e}_t(t) = \dot{x}_r(t) - \dot{x}_f(t) \tag{15}$$

Now, by substituting (5) and (6) on the one hand and (14) and (10) on the other hand into (15), we can obtain:

$$\begin{aligned} \dot{e}_t(t) = \sum_{i,j=1}^{h} \mu_i(\xi(t))\, \mu_j(\xi(t)) \big[& A_{ij}e_t(t) - B_iK_{Ij}Cx_t(t) - B_iK_{Dj}C\dot{e}_t(t) - B_iK_{fj}e_f(t) \\ & + H_{ij}f(t) - R_id(t) \big] \end{aligned} \tag{16}$$

with $A_{ij} = A_i - B_iK_{Pj}C$ and $H_{ij} = B_iK_{fj} - B_i$.

Using the dynamic equation (16), generating $e_t(t)$, the following equality is obtained $\forall i,j = 1,\dots,h$:

$$\begin{gathered} \sum_{i,j=1}^{h} \mu_i(\xi(t))\, \mu_j(\xi(t)) \Delta_{ij}\dot{e}_t(t) = \sum_{i,j=1}^{h} \mu_i(\xi(t))\, \mu_j(\xi(t)) \\ \left[A_{ij}e_t(t) - B_iK_{Ij}Cx_t(t) - B_iK_{fj}e_f(t) + H_{ij}f(t) - R_id(t)\right] \end{gathered} \tag{17}$$

with $\Delta_{ij} = I + B_iK_{Dj}C$.

Under the following Assumption

Assumption 4. $\forall i,j = 1,\dots,h$, the matrices $\Delta_{ij} = I + B_iK_{Dj}C \in \mathbb{R}^{n\times n}$ are invertible; i.e.: $det(\Delta_{ij}) \neq 0$.

Equation (17) can be expressed as follows:

$$\dot{e}_t(t) = \sum_{i,j=1}^{h} \mu_i(\xi(t))\, \mu_j(\xi(t)) \big[\Delta_{ij}{}^{-1} A_{ij} e_t(t) - \Delta_{ij}{}^{-1} B_i K_{Ij} C x_t(t) - \Delta_{ij}{}^{-1} B_i K_{fj} e_f(t) + \Delta_{ij}{}^{-1} H_{ij} f(t) - \Delta_{ij}{}^{-1} R_i d(t) \big] \tag{18}$$

Substituting (8) into $\dot{x}_f(t)$ defined in the state space (6), it may be expressed as follows:

$$\dot{x}_f(t) = \sum_{i,j=1}^{h} \mu_i(\xi(t))\, \mu_j(\xi(t)) \begin{bmatrix} A_i x_f(t) + B_i K_{Pj} C e_t(t) + B_i K_{Ij} C x_t(t) + B_i K_{Dj} C \dot{e}_t(t) \\ +B_i K_{fj} e_f(t) - H_{ij} f(t) + B_i u(t) + R_i d(t) \end{bmatrix} \tag{19}$$

After that, by using (16) and by taking into account the following relationship:

$$B_i K_{Dj} C \Delta_{ij}{}^{-1} = I - \Delta_{ij}{}^{-1} \tag{20}$$

Expression (19) becomes:

$$\dot{x}_f(t) = \sum_{i,j=1}^{h} \mu_i(\xi(t))\, \mu_j(\xi(t)) \begin{bmatrix} A_i x_f(t) + (A_i - \Delta_{ij}{}^{-1} A_{ij}) e_t(t) + \Delta_{ij}{}^{-1} B_i K_{Ij} C x_t(t) \\ +\Delta_{ij}{}^{-1} B_i K_{fj} e_f(t) - \Delta_{ij}{}^{-1} H_{ij} f(t) + B_i u(t) + \Delta_{ij}{}^{-1} R_i d(t) \end{bmatrix} \tag{21}$$

Knowing that $e_s(t) = x_f(t) - \hat{x}_f(t)$ and using the expression of $\hat{x}_f(t)$ in (7), we can write

$$e_s(t) = T_1 x_f(t) - z(t) \tag{22}$$

Since that, for $rank[I_n \quad C]^T = n$, there exists nonsingular matrices $T_1 \in \mathbb{R}^{n\times n}$ and $T_2 \in \mathbb{R}^{n\times m}$ such that

$$[T_1 \quad T_2] \begin{bmatrix} I_n \\ C \end{bmatrix} = I_n \tag{23}$$

Then, a particular solution of (23) using the generalized inverse matrix is given by

$$[T_1 \quad T_2] = \begin{bmatrix} I_n \\ C \end{bmatrix}^+ \tag{24}$$

From (22), the dynamic of the state estimation error obeys the differential equation

$$\dot{e}_s(t) = T_1 \dot{x}_f(t) - \dot{z}(t) \tag{25}$$

Substituting the equations of $\dot{x}_f(t)$ and $\dot{z}(t)$ respectively from (6) and (7), the equation (25) becomes after some calculations

$$\dot{e}_s(t) = \sum_{i=1}^{h} \mu_i(\xi(t)) \big[(T_1 A_i + E_i C - N_i) x_f(t) + (T_1 B_i - G_i) u(t) + N_i e_s(t) + B_i e_f(t) + M_i f(t) + \bar{R}_i d(t) \big] \tag{26}$$

where

$$M_i = T_1 B_i - B_i \tag{27}$$

$$E_i = N_i T_2 - L_i \tag{28}$$

$$\bar{R}_i = T_1 R_i \tag{29}$$

If the following conditions hold $\forall i = 1, \dots, h$:

$$N_i = T_1 A_i + E_i C \tag{30}$$

$$G_i = T_1 B_i \tag{31}$$

The dynamic of the state estimation error reduces to:

$$\dot{e}_s(t) = \sum_{i=1}^{h} \mu_i(\xi(t)) \left[N_i e_s(t) + B_i e_f(t) + M_i f(t) + \bar{R}_i d(t) \right] \tag{32}$$

Finally, the fault estimation error dynamic is defined as follows:

$$\dot{e}_f(t) = \dot{f}(t) - \dot{\hat{f}}(t) \tag{33}$$

3.2. Stability Analysis

To obtain our result, we need the following lemmas:

Lemma1 [18]. For a positive definite matrix Q; that is: $Q = Q^T > 0$ and a positive scalar μ; the following inequality is true:

$$2x^T y \leq \frac{1}{\mu} x^T Q x + \mu y^T Q^{-1} y; \quad x, y \in \mathbb{R}^n \tag{34}$$

Lemma2 [19]. Given real matrices Λ, Y and Z of appropriate dimensions and a matrix J satisfying $J^T J \leq I$, then:

$$\Lambda + YJZ + Z^T J^T Y^T < 0 \tag{35}$$

If and only if there exists a scalar $\eta > 0$ verifying the inequality:

$$\Lambda + \eta Z^T Z + \frac{1}{\eta} Y Y^T < 0 \tag{36}$$

which is equivalent to:

$$\begin{pmatrix} \Lambda & Y & \eta Z^T \\ Y^T & -\eta I & 0 \\ \eta Z & 0 & -\eta I \end{pmatrix} < 0 \tag{37}$$

Now, the goal is to calculate the observer and the controller gains conjointly in one step to ensure the asymptotic convergence of the faulty T-S fuzzy system outputs to the reference model ones. A basic result is summarized in the following theorem.

Theorem 1. Given positive scalars σ, μ, β, and ψ and a positive definite matrix Γ, the PID-FTTC closed loop system ensures the tracking trajectories of the faulty system (6) to the T-S reference model (5) if there exist symmetric and positive definite matrices $X_1 = {P_1}^{-1}$, $X_2 = {P_2}^{-1}$, $X_3 = {P_3}^{-1}$, Q_1, Q_2, Q_3 and Q_4 and matrices $W_{Pj} = K_{Pj} C X_2$, $W_{Ij} = K_{Ij} C X_2$, $W_{Dij} = (I + B_i K_{Dj} C)^{-1}$, $S_i = E_i C X_3$, K_{fi} and Φ_i such that the following LMIs are satisfied for all $i, j, k = 1, \dots, h$:

$$\begin{pmatrix} \bar{X}_{ik} & Y_{ij} & \Psi {Z_{ijk}}^T & P \\ * & -\Psi I & 0 & 0 \\ * & * & -\Psi I & 0 \\ * & * & * & -Q \end{pmatrix} < 0 \tag{38}$$

where

$$\bar{X}_{ik} = \begin{pmatrix} \Pi_i & A_i X_2 & 0 & 0 & 0 \\ * & -\beta I & X_2 & 0 & 0 \\ * & * & -\beta I & 0 & 0 \\ * & * & * & \Theta_i & B_i \\ * & * & * & * & \Sigma_{ik} \end{pmatrix} \tag{39}$$

$$Y_{ij} = \begin{pmatrix} W_{Dij} & 0 & 0 & 0 & 0 \\ -W_{Dij} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\Phi_i C & 0 & 0 \end{pmatrix} \tag{40}$$

$$Z_{ijk}{}^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -X_2 A_i{}^T + W_{Pj}{}^T B_i{}^T & 0 & 0 & 0 & 0 \\ W_{Ij}{}^T B_i{}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & \Xi_k{}^T & 0 & 0 \\ K_{fj}{}^T B_i{}^T & 0 & 0 & 0 & 0 \end{pmatrix} \tag{41}$$

$$P = \begin{pmatrix} X_1{}^T & 0 & 0 & 0 & 0 \\ 0 & X_2{}^T & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & X_3{}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{42}$$

and

$$Q = \begin{pmatrix} \frac{3}{\mu} Q_1 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{\mu} Q_2 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\mu} Q_3 & 0 & 0 \\ 0 & 0 & 0 & \beta I & 0 \\ 0 & 0 & 0 & 0 & \beta I \end{pmatrix} \tag{43}$$

with

$$\Pi_i = A_i X_1 + X_1 A_i{}^T \tag{44}$$

$$\Theta_i = T_1 A_i X_3 + X_3 A_i{}^T T_1{}^T + S_i + S_i{}^T \tag{45}$$

$$\Sigma_{ik} = -\frac{1}{\sigma}\left(\Phi_i C B_k + B_k{}^T C^T \Phi_i{}^T\right) + \frac{3}{\mu\sigma} Q_4 \tag{46}$$

$$\Xi_k = X_3 + \frac{1}{\sigma}(T_1 A_k X_3 + S_k) \tag{47}$$

The controller gain matrices are defined by:

$$K_{Pj} = W_{Pj} P_2 C^{-1} \tag{48}$$

$$K_{Ij} = W_{Ij} P_2 C^{-1} \tag{49}$$

$$K_{Dj} = B_i^{-1}\left({W_{Dij}}^{-1} - I\right)C^{-1} \tag{50}$$

The observer gain matrices can be obtained by:

$$N_i = T_1 A_i + E_i C \tag{51}$$

$$L_i = N_i T_2 - E_i \tag{52}$$

$$G_i = T_1 B_i \tag{53}$$

Proof of Theorem 1. In this proof, we use the theory of Lyapunov to demonstrate the stability of the closed-loop system illustrated in Fig. 1. Then, the problem is turned into an optimization problem through the use of LMI so as to determine the unknown matrices of both the proposed FTC controller and the observer.

Thus, let us consider the following quadratic Lyapunov function:

$$V(\tilde{x}(t)) = \tilde{x}^T(t) P \tilde{x}(t) \tag{54}$$

where $\tilde{x}(t)$ is the augmented state vector defined by (11) and $P = diag\left(P_1, P_2, P_2, P_3, \frac{1}{\sigma}\Gamma^{-1}\right) > 0$ is a positive definite matrix with appropriate dimensions.

The time derivative of the quadratic Lyapunov function (54) leads to

$$\begin{aligned} \dot{V}(\tilde{x}(t)) = {} & {\dot{x}_f}^T(t) P_1 x_f(t) + {x_f}^T(t) P_1 \dot{x}_f(t) + {\dot{e}_t}^T(t) P_2 e_t(t) + {e_t}^T(t) P_2 \dot{e}_t(t) \\ & + {\dot{x}_t}^T(t) P_2 x_t(t) + {x_t}^T(t) P_2 \dot{x}_t(t) + {\dot{e}_s}^T(t) P_3 e_s(t) + {e_s}^T(t) P_3 \dot{e}_s(t) \\ & + \frac{1}{\sigma} {\dot{e}_f}^T(t) \Gamma^{-1} e_f(t) + \frac{1}{\sigma} {e_f}^T(t) \Gamma^{-1} \dot{e}_f(t) \end{aligned} \tag{55}$$

By considering the expressions (21) of $\dot{x}_f(t)$, (18) of $\dot{e}_t(t)$, (12) of $\dot{x}_t(t)$, (32) of $\dot{e}_s(t)$ and (33) of $\dot{e}_f(t)$ and by taking into account the fault estimation of $\dot{\hat{f}}(t)$ in (7), $\dot{V}(\tilde{x}(t))$ becomes

$$\begin{aligned} \dot{V}(\tilde{x}(t)) = \sum_{i,j,k=1}^{h} \mu_{ijk}(\xi(t)) \, [& {x_f}^T(t)(P_1 A_i + {A_i}^T P_1) x_f(t) + 2{x_f}^T(t) P_1 (A_i \\ & - {\Delta_{ij}}^{-1} A_{ij}) e_t(t) + 2{x_f}^T(t) P_1 {\Delta_{ij}}^{-1} B_i K_{Ij} C x_t(t) \\ & + 2{x_f}^T(t) P_1 {\Delta_{ij}}^{-1} B_i K_{fj} e_f(t) + 2{x_f}^T(t) P_1 B_i u(t) \\ & - 2{x_f}^T(t) P_1 {\Delta_{ij}}^{-1} H_{ij} f(t) + 2{x_f}^T(t) P_1 {\Delta_{ij}}^{-1} R_i d(t) + {e_t}^T(t)(P_2 {\Delta_{ij}}^{-1} A_{ij} \\ & + ({\Delta_{ij}}^{-1} A_{ij})^T P_2) e_t(t) + 2{e_t}^T(t)(P_2 - P_2 {\Delta_{ij}}^{-1} B_i K_{Ij} C) x_t(t) \\ & - 2{e_t}^T(t) P_2 {\Delta_{ij}}^{-1} B_i K_{fj} e_f(t) + 2{e_t}^T(t) P_2 {\Delta_{ij}}^{-1} H_{ij} f(t) \\ & - 2{e_t}^T(t) P_2 {\Delta_{ij}}^{-1} R_i d(t) + {e_s}^T(t)(P_3 N_i + {N_i}^T P_3) e_s(t) \\ & + 2{e_s}^T(t) P_3 M_i f(t) + 2{e_s}^T(t)(P_3 B_i - C^T {\Phi_i}^T - \frac{1}{\sigma} {N_k}^T C^T {\Phi_i}^T) e_f(t) \\ & - \frac{2}{\sigma} {e_f}^T(t) \Phi_i C M_k f(t) + 2{e_s}^T(t) P_3 \bar{R}_i d(t) - \frac{1}{\sigma} {e_f}^T(t)(\Phi_i C B_k \\ & + {B_k}^T C^T {\Phi_i}^T) e_f(t) + \frac{2}{\sigma} {e_f}^T(t) \Phi_i C \bar{R}_k d(t) + \frac{2}{\sigma} {e_f}^T(t) \Gamma^{-1} \dot{f}(t)] \end{aligned} \tag{56}$$

Under Assumption 3, we apply Lemma 1 so as to get the following term inequalities from (56):

$$2{x_f}^T(t)P_1B_iu(t) \le \frac{1}{\mu}{x_f}^T(t)Q_1x_f(t) + \eta_{1ij} \tag{57}$$

$$-2{x_f}^T(t)P_1{\Delta_{ij}}^{-1}H_{ij}f(t) \le \frac{1}{\mu}{x_f}^T(t)Q_1x_f(t) + \eta_{2ij} \tag{58}$$

$$2{x_f}^T(t)P_1{\Delta_{ij}}^{-1}R_id(t) \le \frac{1}{\mu}{x_f}^T(t)Q_1x_f(t) + \eta_{3ij} \tag{59}$$

$$2{e_t}^T(t)P_2{\Delta_{ij}}^{-1}H_{ij}f(t) \le \frac{1}{\mu}{e_t}^T(t)Q_2e_t(t) + \eta_{4ij} \tag{60}$$

$$-2{e_t}^T(t)P_2{\Delta_{ij}}^{-1}R_id(t) \le \frac{1}{\mu}{e_t}^T(t)Q_2e_t(t) + \eta_{5ij} \tag{61}$$

$$2{e_s}^T(t)P_3M_if(t) \le \frac{1}{\mu}{e_s}^T(t)Q_3e_s(t) + \eta_{6i} \tag{62}$$

$$2{e_s}^T(t)P_3\bar{R}_id(t) \le \frac{1}{\mu}{e_s}^T(t)Q_3e_s(t) + \eta_{7i} \tag{63}$$

$$-\frac{2}{\sigma}{e_f}^T(t)\Phi_iCM_kf(t) \le \frac{1}{\mu}{e_f}^T(t)Q_4e_f(t) + \eta_{8ik} \tag{64}$$

$$\frac{2}{\sigma}{e_f}^T(t)\Phi_iC\bar{R}_kd(t) \le \frac{1}{\mu}{e_f}^T(t)Q_4e_f(t) + \eta_{9ik} \tag{65}$$

$$\frac{2}{\sigma}{e_f}^T(t)\Gamma^{-1}\dot{f}(t) \le \frac{1}{\mu}{e_f}^T(t)Q_4e_f(t) + \eta_{10} \tag{66}$$

where the scalars $\eta_{1ij}, \eta_{2ij}, \eta_{3ij}, \eta_{4ij}, \eta_{5ij}, \eta_{6i}, \eta_{7i}, \eta_{8ik}, \eta_{9ik}$ and η_{10} are expressed as follows:

$$\eta_{1ij} = \mu\alpha_1^2\lambda_{max}\left(B_i^TP_1Q_1^{-1}P_1B_i\right) \tag{67}$$

$$\eta_{2ij} = \mu\alpha_2^2\lambda_{max}\left(H_{ij}^T\left(\Delta_{ij}^{-1}\right)^TP_1Q_1^{-1}P_1\Delta_{ij}^{-1}H_{ij}\right) \tag{68}$$

$$\eta_{3ij} = \mu\alpha_4^2\lambda_{max}\left(R_i^T\left(\Delta_{ij}^{-1}\right)^TP_1Q_1^{-1}P_1\Delta_{ij}^{-1}R_i\right) \tag{69}$$

$$\eta_{4ij} = \mu\alpha_2^2\lambda_{max}\left(H_{ij}^T\left(\Delta_{ij}^{-1}\right)^TP_2Q_2^{-1}P_2\Delta_{ij}^{-1}H_{ij}\right) \tag{70}$$

$$\eta_{5ij} = \mu\alpha_4^2\lambda_{max}\left(R_i^T\left(\Delta_{ij}^{-1}\right)^TP_2Q_2^{-1}P_2\Delta_{ij}^{-1}R_i\right) \tag{71}$$

$$\eta_{6i} = \mu\alpha_2^2\lambda_{max}(M_i^TP_3Q_3^{-1}P_3M_1) \tag{72}$$

$$\eta_{7i} = \mu\alpha_4^2\lambda_{max}(\bar{R}_i^TP_3Q_3^{-1}P_3\bar{R}_1) \tag{73}$$

$$\eta_{8ik} = \mu\alpha_2^2\lambda_{max}\left(M_k^TC^T\Phi_i^TQ_4^{-1}\Phi_iM_k\right) \tag{74}$$

$$\eta_{9ik} = \mu\alpha_4^2\lambda_{max}\left(\bar{R}_k^T C^T \Phi_i^T Q_4^{-1}\Phi_i C\bar{R}_k\right) \tag{75}$$

$$\eta_{10} = \mu\alpha_2^2\lambda_{max}(\Gamma^{-1T}Q_4^{-1}\Gamma^{-1}) \tag{76}$$

By taking into consideration the inequalities (57)-(66) and the equation (56), $\dot{V}(\tilde{x}(t))$ can be bounded as follows:

$$\begin{aligned}\dot{V}(\tilde{x}(t)) \leq \sum_{i,j,k=1}^{h} \mu_{ijk}(\xi(t))\,[&x_f^T(t)(P_1A_i + A_i^TP_1 + \frac{3}{\mu}Q_1)x_f(t) + 2x_f^T(t)P_1(A_i \\ &-\Delta_{ij}^{-1}A_{ij})e_t(t) + 2x_f^T(t)P_1\Delta_{ij}^{-1}B_iK_{Ij}Cx_t(t) \\ &+2x_f^T(t)P_1\Delta_{ij}^{-1}B_iK_{fj}e_f(t) + e_t^T(t)(P_2\Delta_{ij}^{-1}A_{ij} + \left(\Delta_{ij}^{-1}A_{ij})^TP_2 + \frac{2}{\mu}Q_2\right)e_t(t) \\ &+2e_t^T(t)\left(P_2 - P_2\Delta_{ij}^{-1}B_iK_{Ij}C\right)x_t(t) - 2e_t^T(t)P_2\Delta_{ij}^{-1}B_iK_{fj}e_f(t) \\ &+e_s^T(t)\left(P_3N_i + N_i^TP_3 + \frac{2}{\mu}Q_3\right)e_s(t) + 2e_s^T(t)\left(P_3B_i - C^T\Phi_i^T - \frac{1}{\sigma}N_k^TC^T\Phi_i^T\right)e_f(t) \\ &-\frac{1}{\sigma}e_f^T(t)\left(\Phi_iCB_k + B_k^TC^T\Phi_i^T + \frac{3}{\mu\sigma}Q_4\right)e_f(t)] + \delta\end{aligned} \tag{77}$$

where the scalar δ is the maximum value over i, j, and k such that:

$$\delta = \max_{i,j,k}(\eta_{1ij} + \eta_{2ij} + \eta_{3ij} + \eta_{4ij} + \eta_{5ij} + \eta_{6i} + \eta_{7i} + \eta_{8ik} + \eta_{9ik} + \eta_{10}) \tag{78}$$

At this stage, we can rewrite the inequality (77) under the following reduced form:

$$\dot{V}(\tilde{x}(t)) \leq \tilde{x}^T(t)\left(\sum_{i,j,k=1}^{h} \mu_{ijk}(\xi(t))\Lambda_{ijk}\right)\tilde{x}(t) + \delta \tag{79}$$

where Λ_{ijk} is a matrix defined as follows

$$\Lambda_{ijk} = \begin{pmatrix} \tilde{\Pi}_i & P_1(A_i - \Delta_{ij}^{-1}A_{ij}) & P_1\Delta_{ij}^{-1}B_iK_{Ij}C & 0 & P_1\Delta_{ij}^{-1}B_iK_{fj} \\ * & \tilde{\Omega}_{ij} & P_2 - P_2\Delta_{ij}^{-1}B_iK_{Ij}C & 0 & -P_2\Delta_{ij}^{-1}B_iK_{fj} \\ * & * & 0 & 0 & 0 \\ * & * & * & \tilde{\Theta}_i & \tilde{\Xi}_{ik} \\ * & * & * & * & \Sigma_{ik} \end{pmatrix} \tag{80}$$

with

$$\tilde{\Pi}_i = P_1A_i + A_i^TP_1 + \frac{3}{\mu}Q_1 \tag{81}$$

$$\tilde{\Omega}_{ij} = P_2\Delta_{ij}^{-1}A_{ij} + (\Delta_{ij}^{-1}A_{ij})^TP_2 + \frac{2}{\mu}Q_2 \tag{82}$$

$$\tilde{\Theta}_i = P_3N_i + N_i^TP_3 + \frac{2}{\mu}Q_3 \tag{83}$$

$$\tilde{\Xi}_{ik} = P_3B_i - C^T\Phi_i^T - \frac{1}{\sigma}N_k^TC^T\Phi_i^T \tag{84}$$

$$\Sigma_{ik} = -\frac{1}{\sigma}(\Phi_i C B_k + {B_k}^T C^T {\Phi_i}^T) + \frac{3}{\mu\sigma} Q_4 \tag{85}$$

If the following constraint is satisfied:

$$\sum_{i,j,k=1}^{h} \mu_i(\xi(t))\, \mu_j(\xi(t)) \mu_k(\xi(t)) \Lambda_{ijk} < 0 \tag{86}$$

and if there exists a positive scalar ε defined by:

$$\varepsilon = min\, \lambda \left(- \sum_{i,j,k=1}^{h} \mu_i(\xi(t))\, \mu_j(\xi(t)) \mu_k(\xi(t)) \Lambda_{ijk} \right) < 0 \tag{87}$$

and satisfying the constraint: $\varepsilon \|\tilde{x}(t)\|^2 > \delta;\, \forall t \geq 0$

we deduce that the time derivative of the Lyapunov function $\dot{V}(\tilde{x}(t))$ will be bounded as follows:

$$\dot{V}(\tilde{x}(t)) \leq -\varepsilon \|\tilde{x}(t)\|^2 + \delta \tag{88}$$

Consequently, we can say that $\dot{V}(\tilde{x}(t))$ is negative.

Based on the Lyapunov stability theory, this proves that the augmented system $\tilde{x}(t)$ is stable, meaning that the faulty state vector $x_f(t)$, the state vector $x_t(t)$, the state of the tracking error $e_t(t)$, the state of the estimation error $e_s(t)$, and the fault estimation error $e_f(t)$ are bounded.
By considering the constraint (86), we define a matrix:

$$X = \begin{pmatrix} {P_1}^{-1} & 0 & 0 & 0 & 0 \\ 0 & {P_2}^{-1} & 0 & 0 & 0 \\ 0 & 0 & {P_3}^{-1} & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix} > 0 \tag{89}$$

such that $\Psi_{ijk} = X \Lambda_{ijk} X < 0$.

Then, consider the following change of variables

$$X_1 = {P_1}^{-1} \tag{90}$$

$$X_2 = {P_2}^{-1} \tag{91}$$

$$X_3 = {P_3}^{-1} \tag{92}$$

and computing $\Psi_{ijk} < 0$ that holds

$$\Psi_{ijk} = \begin{pmatrix} X_1 \tilde{\Pi}_i X_1 & (A_i - {\Delta_{ij}}^{-1} A_{ij}) X_2 & {\Delta_{ij}}^{-1} B_i K_{Ij} C X_2 & 0 & {\Delta_{ij}}^{-1} B_i K_{fj} \\ * & X_2 \tilde{\Omega}_{ij} X_2 & X_2 - {\Delta_{ij}}^{-1} B_i K_{Ij} C X_2 & 0 & {\Delta_{ij}}^{-1} B_i K_{fj} \\ * & * & 0 & 0 & 0 \\ * & * & * & X_3 \tilde{\Theta}_i X_3 & X_3 \tilde{\Xi}_{ik} \\ * & * & * & * & \Sigma_{ik} \end{pmatrix} \tag{93}$$

where

$$X_1 \tilde{\Pi}_i X_1 = \Pi_i + \frac{3}{\mu} X_1 Q_1 X_1 \tag{94}$$

$$X_2\tilde{\Omega}_{ij}X_2 = \Omega_{ij} + \frac{2}{\mu}X_2Q_2X_2 \tag{95}$$

$$X_3\tilde{\Theta}_iX_3 = \Theta_i + \frac{2}{\mu}X_3Q_3X_3 \tag{96}$$

$$X_3\tilde{\Xi}_{ik} = B_i - X_3C^T{\Phi_i}^T - \frac{1}{\sigma}X_3{N_k}^TC^T{\Phi_i}^T \tag{97}$$

with

$$\Pi_i = A_iX_1 + X_1{A_i}^T \tag{98}$$

$$\Omega_{ij} = \left({\Delta_{ij}}^{-1}A_{ij}\right)X_2 + X_2\left({\Delta_{ij}}^{-1}A_{ij}\right)^T \tag{99}$$

$$\Theta_i = N_iX_3 + X_3{N_i}^T \tag{100}$$

For all positive scalar $\beta > 0$, replacing the zero in the diagonal of the matrix (93) by the term $\beta I - \beta I = 0$ and using the expression (97) of $X_3\tilde{\Xi}_{ik}$ so as to rewrite (93) this way:

$$\Psi_{ijk} = X_{ik} + Y_{ij}IZ_{ijk} + {Z_{ijk}}^TI{Y_{ij}}^T < 0 \tag{101}$$

where

$$X_{ik} = \begin{pmatrix} X_1\tilde{\Pi}_iX_1 & A_iX_2 & 0 & 0 & 0 \\ * & \frac{2}{\mu}X_2Q_2X_2 & X_2 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & X_3\tilde{\Theta}_iX_3 & B_i \\ * & * & * & * & \Sigma_{ik} \end{pmatrix} \tag{102}$$

$$Y_{ij} = \begin{pmatrix} {\Delta_{ij}}^{-1} & 0 & 0 & 0 & 0 \\ -{\Delta_{ij}}^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\Phi_iC & 0 & 0 \end{pmatrix} \tag{103}$$

$$Z_{ijk} = \begin{pmatrix} 0 & -A_{ij}X_2 & B_iK_{Ij}CX_2 & 0 & B_iK_{fj} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Xi_k & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{104}$$

with

$$\Xi_k = X_3 + \frac{1}{\sigma}N_kX_3 \tag{105}$$

By applying the Lemma 2, the inequality (101) is verified if and only if there exists a scalar $\psi > 0$ satisfying

$$X_{ik} + \psi{Z_{ijk}}^TZ_{ijk} + \frac{1}{\psi}{Y_{ij}}^TY_{ij} < 0 \tag{106}$$

That may be equivalent to

$$\begin{pmatrix} X_{ik} & Y_{ij} & \psi {Z_{ijk}}^T \\ * & -\psi I & 0 \\ * & * & -\psi I \end{pmatrix} < 0 \tag{107}$$

Now, we dissociate the term $\frac{3}{\mu} X_1 Q_1 X_1$ from the inequality (107), which may be reformulated as follows

$$\begin{pmatrix} \tilde{X}_{ik} & Y_{ij} & \psi {Z_{ijk}}^T \\ * & -\psi I & 0 \\ * & * & -\psi I \end{pmatrix} - {\bar{X}_1}^T \left(-\frac{3}{\mu} Q_1\right) \bar{X}_1 < 0 \tag{108}$$

where

$$\tilde{X}_{ik} = \begin{pmatrix} \Pi_i & A_i X_2 & 0 & 0 & 0 \\ * & \frac{2}{\mu} X_2 Q_2 X_2 & X_2 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & X_3 \tilde{\Theta}_i X_3 & B_i \\ * & * & * & * & \Sigma_{ik} \end{pmatrix} \tag{109}$$

and

$$\bar{X}_1 = (X_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0) \tag{110}$$

Now, we apply the modified Schur Lemma in the above inequality (108), and then we repeat the same steps successively for the terms $\frac{2}{\mu} X_2 Q_2 X_2$ and $\frac{2}{\mu} X_3 Q_3 X_3$ in order to get:

$$\begin{pmatrix} \hat{X}_{ik} & Y_{ij} & \psi {Z_{ijk}}^T & \hat{P} \\ * & -\psi I & 0 & 0 \\ * & * & -\psi I & 0 \\ * & * & * & -\hat{Q} \end{pmatrix} < 0 \tag{111}$$

where

$$\hat{X}_{ik} = \begin{pmatrix} \Pi_i & A_i X_2 & 0 & 0 & 0 \\ * & 0 & X_2 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & \Theta_i & B_i \\ * & * & * & * & \Sigma_{ik} \end{pmatrix} \tag{112}$$

$$\hat{P} = \begin{pmatrix} {X_1}^T & 0 & 0 \\ 0 & {X_2}^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & {X_3}^T \\ 0 & 0 & 0 \end{pmatrix} \tag{113}$$

and

$$\hat{Q} = \begin{pmatrix} \frac{3}{\mu}Q_1 & 0 & 0 \\ 0 & \frac{2}{\mu}Q_2 & 0 \\ 0 & 0 & \frac{2}{\mu}Q_3 \end{pmatrix} \tag{114}$$

In the following, we replace the two zero in the diagonal of the matrix $\hat{X}_{ik}$ (112) by the term $\beta I - \beta I = 0$ for all positive scalar $\beta > 0$. After that, we dissociate the terms $\beta I = I(\beta I)I$ successively from the inequality (111) in a similar way to (108), and we use the modified Schur Lemma so as to rewrite (111) as follows:

$$\begin{pmatrix} \hat{X}_{ik} & Y_{ij} & \psi {Z_{ijk}}^T & \hat{P} \\ * & -\psi I & 0 & 0 \\ * & * & -\psi I & 0 \\ * & * & * & -\hat{Q} \end{pmatrix} < 0 \tag{115}$$

where

$$\hat{X}_{ik} = \begin{pmatrix} \Pi_i & A_i X_2 & 0 & 0 & 0 \\ * & -\beta I & X_2 & 0 & 0 \\ * & * & -\beta I & 0 & 0 \\ * & * & * & \Theta_i & B_i \\ * & * & * & * & \Sigma_{ik} \end{pmatrix} \tag{116}$$

$$\hat{P} = \begin{pmatrix} {X_1}^T & 0 & 0 & 0 & 0 \\ 0 & {X_2}^T & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & {X_3}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{117}$$

and

$$\hat{Q} = \begin{pmatrix} \frac{3}{\mu}Q_1 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{\mu}Q_2 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{\mu}Q_3 & 0 & 0 \\ 0 & 0 & 0 & \beta I & 0 \\ 0 & 0 & 0 & 0 & \beta I \end{pmatrix} \tag{118}$$

Using the expressions A_{ij} and N_i and taking into account the following change of variables:

$$W_{Pj} = K_{Pj} C X_2 \tag{119}$$

$$W_{Ij} = K_{Ij} C X_2 \tag{120}$$

$$W_{Dij} = (I + B_i K_{Dj} C)^{-1} \tag{121}$$

$$S_i = E_i C X_3 \tag{122}$$

The inequality (115) becomes reformulated as a Linear Matricial Inequality (LMI), which can be rewritten as in **Theorem 1**. The proof of the theorem is now completed.

In order to solve numerically **Theorem 1**, it is important to proceed with the following algorithm:

Algorithm 1: PID-FTTC closed loop system design

For all $i, j, k = 1, \dots, h$

- Compute the matrices T_1 and T_2 from (24).
- Give values for the positive scalars β, σ, μ, and ψ and the positive definite matrix Γ.
- Find solutions for the unknown matrices:X_1, X_2, X_3, Q_1, Q_2, Q_3, W_{Pj}, W_{Ij}, W_{Dij}, S_i, K_{fj} and Φ_i by solving the LMIs (38) in Theorem 1.
- Compute the matrices E_i by: $E_i = S_i(CX_3)^{-1}$
- Compute the gains matrices of the PID-FTTC and AFO: K_{Pj}, K_{Ij}, K_{Dj}, N_i, L_i and G_i from (48)-(53).

end

Remark 1: From the steps of computation in Algorithm 1, we see that the main computation of the designed method is to solve the optimization problem (38) except for some simple matrix calculations. Meanwhile, the solution of (38) can be easily obtained by the MATLAB LMI toolbox, which means that the observer gain and the PID-FTTC law matrices can be computed by using the expressions (48)-(53). So, the computation complexity of the results is very small.

4. Illustrative Example: Single-Link Flexible Joint Robot

In this part, we illustrate the proposed approach, which concerns the T-S fuzzy systems affected by actuator faults and unknown bounded disturbances. For this goal, a simulation example is carried out for a single-link flexible joint robot, as it is shown in Fig. 2.

4.1. Takagi-Sugeno Model Design

The nonlinear model of the single-link flexible joint robot arm may be represented by the following set of equations [20]

$$\begin{cases} \dot{\theta}_m(t) = \omega_m(t) \\ \dot{\omega}_m(t) = \dfrac{k}{J_m}(\theta_l(t) - \theta_m(t)) - \dfrac{B_v}{J_m}\omega_m(t) + \dfrac{K_\tau}{J_m}u(t) \\ \dot{\theta}_l(t) = \omega_l(t) \\ \dot{\omega}_l(t) = -\dfrac{k}{J_l}(\theta_l(t) - \theta_m(t)) - \dfrac{mgh}{J_l}sin(\theta_l(t)) \end{cases} \tag{123}$$

where, J_m represents the inertia of the motor, J_l is the inertia of the controlled link, m is the link mass, h is the center of mass, g is the acceleration due to gravity, k is the elastic constant, B_v is the viscous friction coefficient and K_τ is the amplifier gain. The numerical values of these parameters are listed in Table 1 [8].

Table 1. Parameters for the single-link flexible joint robot model

Parameter	Meaning	Value
J_m	Motor inertia	$3.7 \times 10^{-3} Kg.m^2$
J_l	Link inertia	$9.3 \times 10^{-2} Kg.m^2$
h	Link length	$1.55 \times 10^{-2} m$
m	Pointer mass	$2.04 \times 10^{-1} Kg$
k	Torsional spring constant	$1.8 \times 10^{-1} N.m.rad^{-1}$
B_v	Viscous friction coefficient	$4.6 \times 10^{-3} N.m.V^{-1}$
K_τ	Amplifier gain	$8 \times 10^{-2} N.m.V^{-1}$

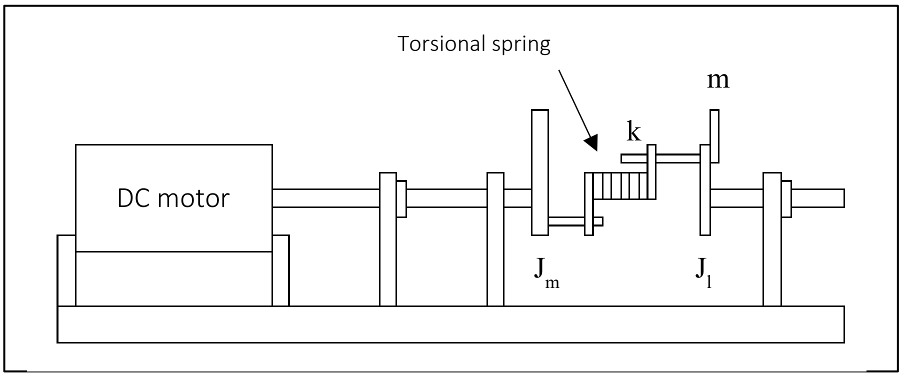

Fig. 2. Single-link flexible manipulator arm actuated by DC motor.

The state vector and the output vector are defined respectively by $x(t) = [x_1^T \quad x_2^T \quad x_3^T \quad x_4^T]^T = [\theta_m^T \quad \omega_m^T \quad \theta_l^T \quad \omega_l^T]^T$ and $y(t) = [\theta_m^T \quad \omega_m^T]^T$ where $x_1(t)$ stands for the angular rotation of the motor, $x_2(t)$ is the angular velocity of the motor, $x_3(t)$ is the angular position of the link and $x_4(t)$ is the angular velocity of the link. The system in (123) can be represented as follows:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{124}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{J_m} & -\frac{B_v}{J_m} & \frac{k}{J_m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_l} & 0 & -\frac{k}{J_l} - \frac{mgh}{J_l}\frac{sin(x_3)}{x_3} & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{K_\tau}{J_m} \\ 0 \\ 0 \end{bmatrix}$$

To build the T-S fuzzy model for the single-link flexible joint robot, the following premise variable is considered:

$$\xi(t) = \frac{sin(x_3)}{x_3}$$

such that $x_3 \in \left[-\frac{\pi}{2}, 0\right]$

where the number of nonlinearities is $n = 1$, the global model can be represented by $r = 2^n = 2$ sub-models.

The local membership functions are defined by:

$$F_1^1(\xi(t)) = \frac{\xi(t)-\underline{\xi}}{\bar{\xi}-\underline{\xi}} \quad \text{and} \quad F_1^2(\xi(t)) = \frac{\bar{\xi}-\xi(t)}{\bar{\xi}-\underline{\xi}}$$

where the scalars $\underline{\xi}$ and $\bar{\xi}$ are:

$$\begin{cases} \underline{\xi} = \min_{x,u} \xi(x_2(t)) \\ \bar{\xi} = \max_{x,u} \xi(x_2(t)) \end{cases}$$

Finally, the activation functions are:

$$\mu_1(\xi(t)) = F_1^1(\xi(t)) \quad \text{and} \quad \mu_2(\xi(t)) = F_1^2(\xi(t))$$

Then, the single-link flexible joint robot system (123) is represented by the T-S fuzzy model:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} \mu_i(\xi(t))\,(A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{cases} \tag{125}$$

This system is considered to be affected by actuator fault and external disturbance. So, the model (125) can be rewritten as follows:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{2} \mu_i(\xi(t))(A_i x(t) + B_i(u(t) + f(t)) + R_i d(t)) \\ y(t) = Cx(t) \end{cases} \tag{126}$$

The computed matrices A_i, B_i, R_i and C of each sub-model is given as follows:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 1.95 & 0 & -2.28 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 1.95 & 0 & -2.16 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 = R_2 = \begin{bmatrix} 0.02 \\ 0.01 \\ 0.02 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

4.2. Simulation Results

The proposed PID-FTTC based on an adaptive fuzzy observer for the T-S fuzzy system (126) affected by additive actuator fault and unknown input is designed by solving the LMI problem given into **Theorem 1** under MATLAB LMI Toolbox. The parameter values are chosen to $\sigma = 2$, $\mu = 1$, $\beta = 2$, $\psi = 1$, and $\Gamma = 1$.

Solving the LMI (38), we get a feasible solution that gives the following matrices of the observer (7) and of the PID-FTTC law (8) that are computed by using the expressions (48)-(53):

$$N_1 = \begin{bmatrix} -1.9962 & 0.0000 & 0.0000 & -0.0000 \\ -0.0000 & -1.9959 & 0.0000 & 0.0000 \\ 0.0000 & -0.0000 & -1.9962 & 0.0000 \\ -0.0000 & 0.0000 & -0.0000 & -1.9962 \end{bmatrix}$$

$$N_2 = \begin{bmatrix} -1.9959 & 0.0000 & 0.0000 & -0.0000 \\ -0.0000 & -1.9958 & 0.0000 & 0.0000 \\ 0.0000 & -0.0000 & -1.9959 & 0.0000 \\ -0.0000 & 0.0000 & -0.0000 & -1.9959 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 0.9981 & 0.5000 & -0.0000 & 0.0000 \\ -24.3000 & 0.3730 & 24.3000 & -0.0000 \\ -0.0000 & 0.0000 & 0.9981 & 0.5000 \\ 0.9750 & -0.0000 & -1.1400 & 0.9981 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 0.9980 & 0.5000 & -0.0000 & 0.0000 \\ -24.3000 & 0.3729 & 24.3000 & -0.0000 \\ -0.0000 & 0.0000 & 0.9980 & 0.5000 \\ 0.9750 & -0.0000 & -1.0800 & 0.9980 \end{bmatrix}$$

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$$G_1 = G_2 = [0 \quad 10.8 \quad 0 \quad 0]^T$$
$$\Phi_1 = [0 \quad 17.4156 \quad 0 \quad 0]$$
$$\Phi_2 = [0 \quad 17.7594 \quad 0 \quad 0]$$
$$K_{P1} = [3.3281 \quad 0.8310 \quad \text{-}1.5013 \quad 3.0854]$$
$$K_{P2} = [3.3004 \quad 0.8133 \quad \text{-}1.4455 \quad 2.9962]$$
$$K_{I1} = [\text{-}0.0196 \quad 0.0179 \quad 0.0225 \quad 0.0234]$$
$$K_{I1} = [\text{-}0.0196 \quad 0.0179 \quad 0.0225 \quad 0.0234]$$
$$K_{D1} = [\text{-}0.1793 \quad \text{-}0.0042 \quad 0.0432 \quad \text{-}0.0033]$$
$$K_{D2} = [\text{-}0.1781 \quad \text{-}0.0039 \quad 0.0432 \quad \text{-}0.0029]$$
$$K_{f1} = 4.7715$$
$$K_{f2} = 4.4014$$

Let us consider the actuator fault $f(t)$ affecting the system can be expressed as follows:

$$f(t) = \begin{cases} 10^{-2}(-0.05t + 1.5) & \forall t \in [10,20s] \\ 10^{-2}(2 + sin(0.1t)) & \forall t \in [30,45s] \end{cases}$$

and $u(t) = 0.1\, sin(1.5t)$

To illustrate the robustness of the observer-based tracking FTC design, the disturbance input $d(t)$ is assumed to be a Gaussian distributed random signal with zero mean and unit variance. The simulation results are presented in the following figures. Fig. 3 illustrates the actuator fault signal $f(t)$ and its estimate $\hat{f}(t)$. It can be seen that, the fault estimation $\hat{f}(t)$ converges to the real fault $f(t)$ accurately. From this figure, we note the ability of the proposed AFO to provide a good estimation quality of actuator faults despite the presence of disturbances.

The fault-tolerant tracking control law is compared to the nominal control input and illustrated in Fig. 4. In Figs. 5-7, we present the comparisons of the faulty system outputs in the case without PID-FTTC and the case with PID-FTTC. We can see that without PID-FTTC law, the system outputs deviate from their reference. However, the proposed PID-FTTC controllers can preserve the tracking performances and ensure excellent fault-tolerant performances.

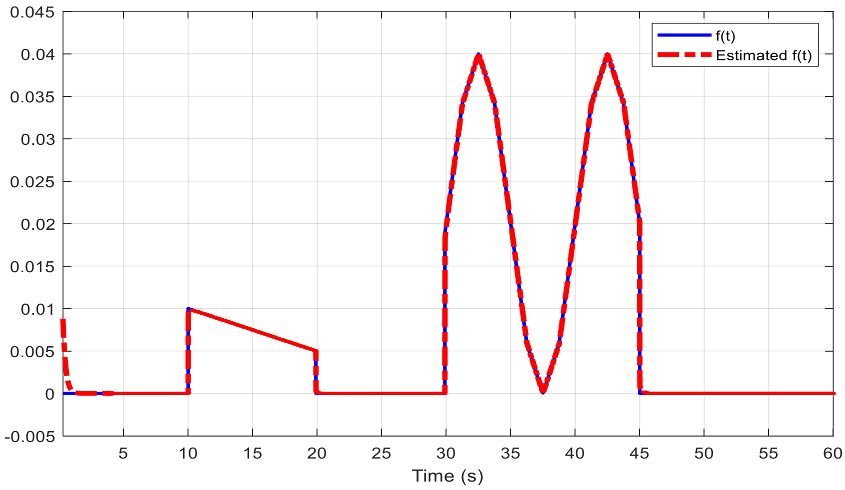

Fig. 3. Actuator fault $f(t)$ and its estimate $\hat{f}(t)$

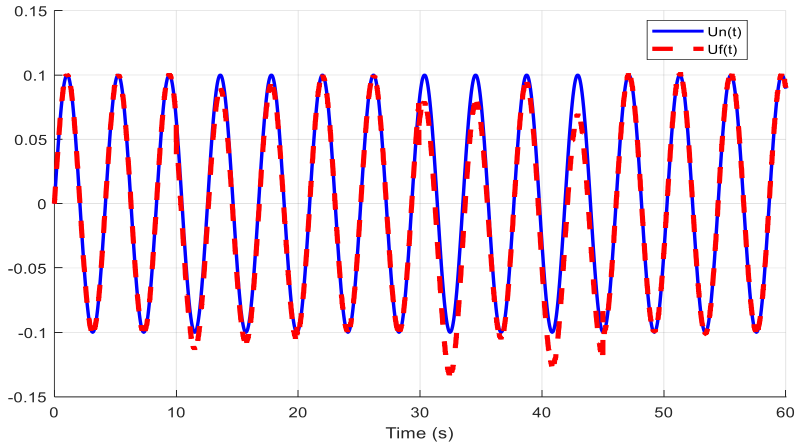

Fig. 4. Nominal and PID-FTTC signals

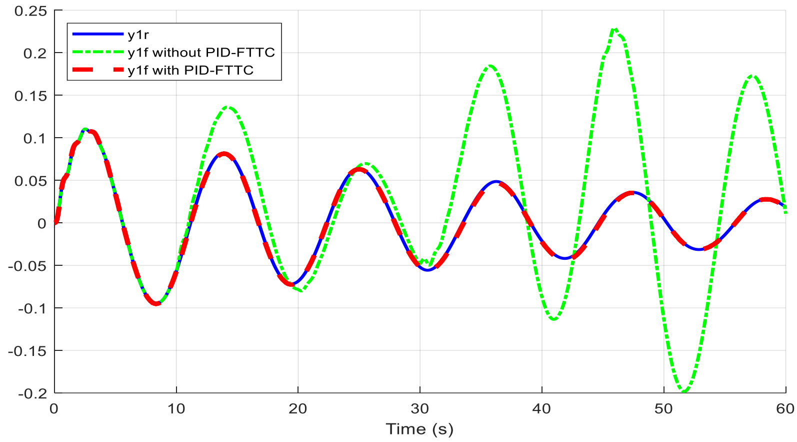

Fig. 5. Comparison of output $y_1(t)$ in the following cases: without fault, with fault without PID-FTTC, with fault and PID-FTTC

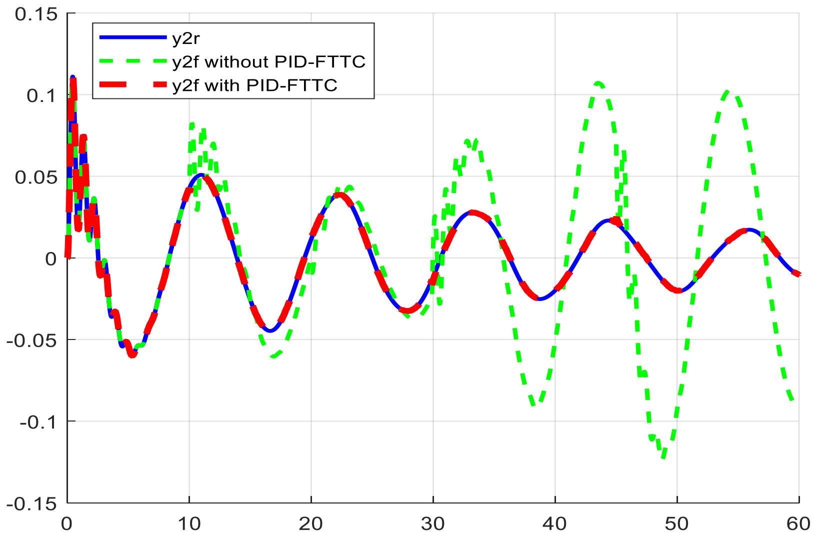

Fig. 6. Comparison of output $y_2(t)$ in the following cases: without fault, with fault without PID-FTTC, with fault and PID-FTTC

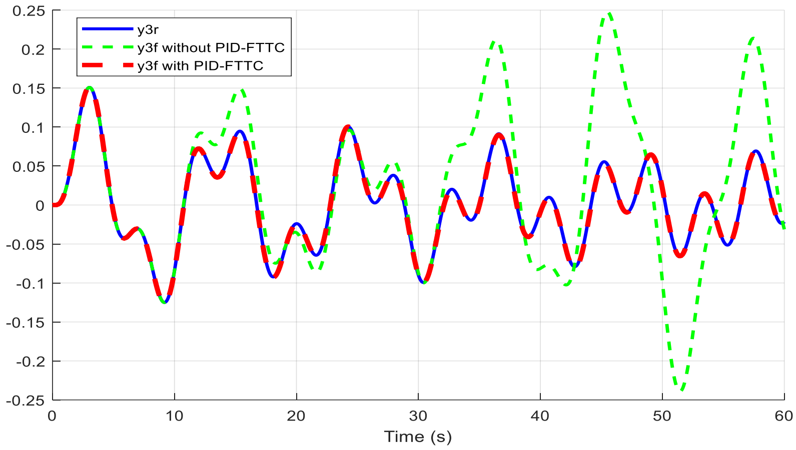

Fig. 7. Comparison of output $y_3(t)$ in the following cases: without fault, with fault without PID-FTTC, with fault and PID-FTTC

The state estimation errors are shown in Fig. 8, from which we can see that the proposed Adaptive Fuzzy Observer can estimate the states very accurately, even after the occurrence of an actuator fault. Through the above simulation, we can conclude that the proposed approach can achieve the objects of actuator fault estimation successfully, and it can compensate for the actuator fault effects and ensures the tracking between the reference model outputs and the faulty system outputs.

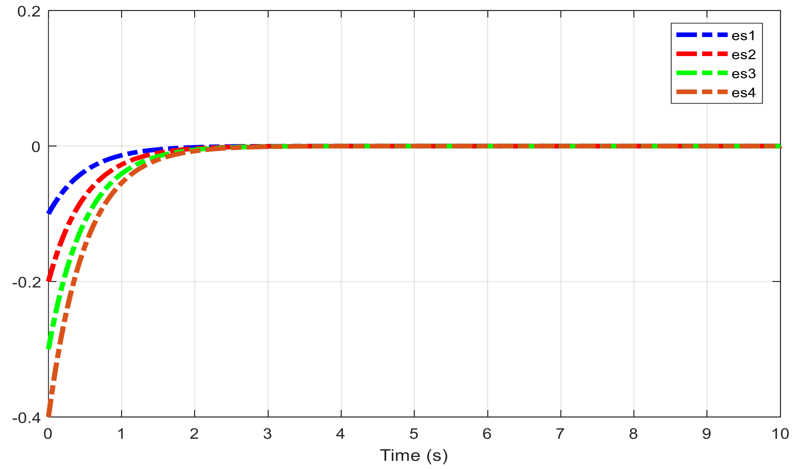

Fig. 8. Errors between faulty states and their estimates

5. Conclusion

This paper proposes an AFO which is able to diagnose and estimates actuator faults affecting nonlinear systems described by T-S fuzzy models in the presence of unknown bounded disturbances. This observer-based FTC has been designed in order to ensure the stability convergence of the closed loop system and the tracking between the faulty system and a healthy reference model. For the stabilization, sufficient conditions are formulated in terms of LMIs using the appropriate Lyapunov function. Moreover, the controller and the AFO gains have been computed using a single-step procedure by solving a set of LMIs, reducing the conservatism. Finally, the proposed results are then applied to a single-link flexible joint robot subject to external disturbances with additive actuator faults. Simulation results show that the developed strategy was able to compensate for actuator faults and ensure some performances such as stability, rapidity, accuracy, and model reference tracking in spite of the existence of unknown inputs. In future work, this proposed method will be extended to investigate the FTC design problem for T-S fuzzy nonlinear systems in the presence of both actuator faults and sensor faults.

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