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Global Existence for Heat Equation with Nonlinear and Damping Piecewise Neumann Boundary Condition

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ABSTRACT

The Columbia space shuttle catastrophe in 2003 served as the inspiration for this paper's improved mathematical model, which includes a nonlinear damping Neumann boundary condition. By creating and examining a modified heat equation with piecewise nonlinear source terms and damping Neumann boundary conditions, the study seeks to investigate the incident's heat transport dynamics. To ensure that the problem is well-posed, we provide strong mathematical arguments for the existence of solutions both locally and globally. In addition, we use numerical simulations to show how the nonlinear boundary conditions affect heat dissipation and to confirm the theoretical results. The findings advance our knowledge of thermal modeling in aircraft applications and offer greater insights into heat propagation under such conditions.

1. Introduction

The study of partial differential equations (PDEs) has been a key focus in mathematical analysis, particularly in understanding existence, uniqueness, and stability of solutions across various applications [1]–[4]. While our work primarily examines a nonlinear heat equation with damping Neumann boundary conditions, fractional differential equations have been widely explored for their ability to model memory-dependent processes in physics and engineering [5]–[11]. Recent advances in reaction-diffusion problems and boundary-value analysis highlight the significance of nonlinear terms in influencing solution behavior, including blow-up and stabilization phenomena [12]–[18]. Techniques such as energy estimates and integral boundary conditions have been applied to various nonlinear PDEs, ensuring well-posedness and guiding numerical approximations [19]–[26]. Although fractional calculus is not the central theme of this study, its methodologies, including the Faedo-Galerkin method and energy-based approaches, remain crucial in addressing complex systems governed by nonlocal operators [27]–[30].

Thermal insulation foam was applied to the shuttle's primary fuel tank to stop ice from forming when liquid hydrogen and oxygen were added. Unfortunately, on February 1, 2003, the shuttle's re-entry into Earth's atmosphere resulted in the Columbia disaster. The orbiter's left wing was struck by a piece of foam insulation that had broken off from the external tank, causing the catastrophe. In particular, a piece of foam about the size of a suitcase broke off and struck the Columbia's left wing's reinforced carbon-carbon (RCC) panels. According to ground tests by the Columbia Accident Investigation Board, this incident most likely created a hole that was six to ten inches (15 to 25 cm) in diameter, which would have allowed hot gasses to penetrate the wing upon re-entry.

Subsequently, a mathematical analysis was conducted to investigate the heat transfer mechanisms associated with the breach in the shuttle's wing, which resulted from the foam impact. To assess the thermal behavior during re-entry, comprehensive simulations and numerical computations were carried out, incorporating various physical and chemical phenomena shown in Fig. 1.

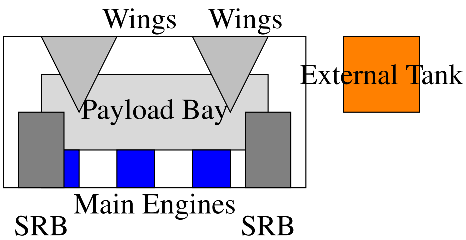

Fig. 1. Space Shuttle Columbia

The study specifically focused on the role of radiative heat flux, which was identified as the dominant factor in heat propagation through the damaged section of the left wing Fig. 2. The heat transfer model considered the interactions between conduction, convection, and radiation, along with the impact of high-temperature gas dynamics and material degradation processes. These elements were integrated into the mathematical framework to accurately represent the thermal response of the shuttle under extreme aerodynamic heating conditions [31].

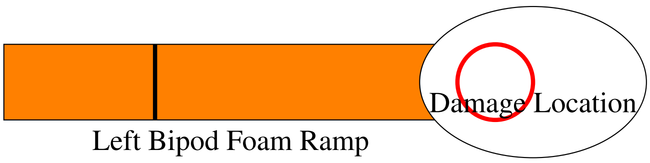

Fig. 2. Left Bipod Foam Ramp

The Columbia disaster prompted significant reforms in the space shuttle program, prioritizing enhanced safety measures and risk mitigation strategies. One of the most critical improvements was the implementation of rigorous inspection protocols to thoroughly evaluate the shuttle's Thermal Protection System (TPS) before and during missions. Particular emphasis was placed on detailed assessments of high-risk components, such as the Reinforced Carbon-Carbon (RCC) panels, which play a crucial role in shielding the shuttle from extreme re-entry temperatures.

To further mitigate risks, in-orbit inspection capabilities were introduced, allowing astronauts to assess the structural integrity of the TPS while in space. These inspections were facilitated by advanced imaging technologies and robotic systems, enabling the identification of potential damage that could compromise mission safety. Additionally, extensive research and development efforts were undertaken to devise effective on-orbit repair techniques. Astronauts were equipped with specialized tools and innovative repair materials designed to address specific types of TPS damage encountered during missions. These advancements significantly bolstered the resilience of space shuttle operations and contributed to improved safety standards in future space exploration endeavors [31].

Motivated by these developments, we were inspired to develop a mathematical model [32]–[35] of the following form:

$$\begin{cases} u_t - \Delta u = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,0) = \varphi(x), & x \in \Omega, \\ \frac{\partial u}{\partial \eta} = 0, & (x,t) \in \Gamma_0 \times (0,T), \\ \frac{\partial u}{\partial \eta} = -\left|u_t\right|^{q-2} u_t + u^p, & (x,t) \in \Gamma_1 \times (0,T). \end{cases} \tag{PP}$$

The boundary conditions on Γ_0 and Γ_1 in the context of heat transfer typically represent different types of thermal interactions at the system boundaries. Specifically, Γ_0 denotes a boundary where heat is neither entering nor leaving the system, often referred to as an adiabatic boundary.

$$\frac{\partial u}{\partial \eta} = 0$$

The condition implies that there is no heat flux normal to the boundary, ensuring that the temperature gradient in the outward normal direction is zero. Consequently, no heat transfer occurs through Γ_0. On the other hand, Γ_1 represents a boundary where heat exchange with the surroundings takes place.

$$\frac{\partial u}{\partial \eta} = -\left|u_t\right|^{q-2} u_t + u^p$$

The boundary condition defines the heat flux at Γ_1. The term $-\left|u_t\right|^{q-2} u_t$ characterizes the convective heat transfer, which depends on the magnitude and direction of the temperature gradient $\frac{\partial u}{\partial \eta}$ at the boundary, acting as a damping term. Meanwhile, the term u^p represents an external heat source, contributing to the heat distribution in the system.

The values of q and p in the boundary condition equation play a significant role in determining heat transfer behavior at the boundary. Specifically, q influences the dependency of the convective heat transfer term on the magnitude of the temperature gradient $\frac{\partial u}{\partial \eta}$ at the boundary. The term $\left|u_t\right|^{q-2} u_t$ represents convective heat transfer, where the exponent $\left|u_t\right|^{q-2}$ regulates how the magnitude of the temperature gradient affects the overall heat flux. Meanwhile, the parameter p determines the contribution of the temperature distribution within the system to the heat transfer process at the boundary. The term u^p serves as a heat source, influencing the rate of heat exchange at Γ_1 and altering the thermal behavior of the system [36]–[38].

The organization of this paper is as follows: Section 2 is dedicated to establishing the existence of the linear problem u. In Section 3, we present the existence of the main problem along with an analysis of blow-up phenomena and global existence. Finally, in the last section, we discuss numerical simulations of solutions for the main problem.

2. Existence of Heat Equation

In this section, we present the heat equation with piecewise nonlinear dynamical boundary conditions. We begin with the formulation of the problem (P_1) and subsequently analyze its existence. This section is particularly significant as it focuses on establishing the existence of a solution using the Faedo-Galerkin method.

2.1. Formulation of the Problem

In this subsection, we define the mathematical model governing the heat equation with piecewise nonlinear boundary conditions. The problem is formulated as follows:

$$\begin{cases} u_t - \Delta u = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,0) = \varphi(x), & x \in \Omega, \\ \frac{\partial u}{\partial \eta} = 0, & (x,t) \in \Gamma_0 \times (0,T), \\ \frac{\partial u}{\partial \eta} = -\left|u_t\right|^{q-2} u_t + h(x,t), & (x,t) \in \Gamma_1 \times (0,T). \end{cases} \tag{P$_1$}$$

where $u = u(x,t)$, with $t \geq 0$ and $x \in \Omega$. The domain Ω is a bounded open subset of $\mathbb{R}^n$ $(n \geq 1)$ with a regular boundary $\partial\Omega$, satisfying $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Here, g represents a given forcing term acting on Γ_1.

2.2. Results

In this subsection, we present key theoretical results related to the existence and properties of weak solutions to the heat equation with nonlinear boundary conditions. We begin with a fundamental lemma that establishes the necessary function spaces and integral formulations.

Lemma 2.1 Let $\xi \in L^{q'}((0,T) \times \Gamma_1)$, and suppose that u is a weak solution of the problem

$$\begin{cases} u_t - \Delta u = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,0) = \varphi(x), & x \in \Omega, \\ \frac{\partial u}{\partial \eta} = 0, & (x,t) \in \Gamma_0 \times (0,T), \\ \frac{\partial u}{\partial \eta} = \xi, & (x,t) \in \Gamma_1 \times (0,T). \end{cases}$$

That is, u satisfies

$$u \in L^\infty\left(0,T; H^1_{\Gamma_0}(\Omega)\right),$$

such that

$$u_t \in L^2(0,T; L^2(\Omega)) \cap L^m((0,T) \times \Gamma_1),$$

and for all $\phi \in H^1_{\Gamma_0}(\Omega) \cap L^m(\Gamma_1)$ and $\phi \in C_c((0,T); H^1(\Omega))$, we have

$$\int_\Omega u_t \phi + \int_\Omega \nabla u \cdot \nabla \phi = \int_{\Gamma_1} \xi \phi = 0.$$

Then, we have the regularity result

$$u \in C([0,T]; H^1(\Omega)).$$

Moreover, the following energy identity holds for $0 \leq s \leq t \leq T$:

$$\frac{1}{2}\left\|\nabla u(t)\right\|_2^2 - \frac{1}{2}\left\|\nabla u(s)\right\|_2^2 + \int_s^t \left\|u_t\right\|_2^2 \, dt = \int_s^t \int_{\Gamma_1} \xi u_t \, dx dt.$$

Proof 1 Let $F = L^2(\Omega)$, $E = H^1(\Omega)$, and $B = L^m(\Gamma_1)$. To prove the existence and uniqueness of the solution to problem (P_1), we consider a sequence $(w_n)_n$ in the space $H^1(\Omega) \cap L^m(\Gamma_1)$. These vectors are linearly independent, and every finite combination forms a dense subset in $H^1(\Omega) \cap L^m(\Gamma_1)$. Moreover, they are orthonormal in $L^2(\Omega)$. Let $u_{0n} \in \text{span}\{w_0, ..., w_n\}$ such that $u_{0n} \to u_0$ in $L^2(\Omega)$.

Lemma 2.2 Let Ω be a bounded and regular domain in $\mathbb{R}^n$, where Γ_1 is also bounded. Then, the space $H^1(\Omega) \cap L^m(\Gamma_1)$ is dense in $H^1(\Omega)$.

Proof 2 It is sufficient to prove that $H^1(\Omega) \cap L^\infty(\Gamma_1)$ is dense in $H^1(\Omega)$. Let $u \in H^1(\Omega)$, and define the truncated sequence $(u_n)_{n\in\mathbb{N}}$ as follows:

$$u_n = n - \left[2n - (u+n)^+\right]^+ = \begin{cases} u, & \text{if } |u| \le n, \\ \frac{nu}{|u|}, & \text{if } |u| > n. \end{cases}$$

Then, by applying the following lemma, we obtain the desired result.

Lemma 2.3 Let $u \in W^1(\Omega)$. Then, the functions u^+, u^-, and $|u|$ belong to $W^1(\Omega)$, and their derivatives satisfy

$$Du^+ = \begin{cases} Du, & \text{if } u > 0, \\ 0, & \text{if } u \le 0. \end{cases}$$

$$Du^- = \begin{cases} 0, & \text{if } u > 0, \\ -Du, & \text{if } u \le 0. \end{cases}$$

$$D|u| = \begin{cases} Du, & \text{if } u > 0, \\ 0, & \text{if } u = 0, \\ -Du, & \text{if } u < 0. \end{cases}$$

Proof 3 For $\varepsilon > 0$, define the function

$$f_\varepsilon(u) = \begin{cases} (u^2+\varepsilon^2)^{\frac{1}{2}} - \varepsilon, & \text{if } u > 0, \\ 0, & \text{if } u \le 0. \end{cases}$$

By applying Lemma 7.5 (from [39], [40]): Let $f \in C^1(\mathbb{R})$, with $f' \in L^\infty(\mathbb{R})$ and let $u \in W^1(\Omega)$. Then, the composite function $f \circ u$ belongs to $W^1(\Omega)$ and satisfies

$$D(f \circ u) = f'(u)Du.$$

Applying this result, for any test function $\varphi \in C_0^1(\Omega)$, we obtain

$$\int_\Omega f_\varepsilon(u) D\varphi\, dx = -\int_{u>0} \frac{uDu}{(u^2+\varepsilon^2)^{\frac{1}{2}}}\, dx.$$

Taking the limit as $\varepsilon \to 0$, we obtain

$$\int_\Omega u^+ D\varphi\, dx = -\int_{u>0} \varphi Du\, dx.$$

Thus, we have established that $Du^+, Du^-, D|u|$ exist for u^+. The remaining results follow directly since

$$u^- = -(-u)^+, \quad |u| = u^+ - u^-.$$

In what follow, we construct approximate solutions for the main problem using a Galerkin-type method. By considering a suitable basis in the function space and applying truncation techniques, we obtain a sequence of finite-dimensional approximations that satisfy energy estimates. These estimates play a crucial role in proving the existence of weak solutions in the Sobolev space framework.

We further employ compactness arguments and monotonicity techniques to ensure convergence and derive the necessary regularity properties. To this end, we have the boundary property

$$u^+\big|_{\partial\Omega} = \left(u|_{\partial\Omega}\right)^+, \quad \forall u \in H^1(\Omega).$$

Moreover, the inequality

$$\|v\|_{L^\infty(\Gamma_1)} \leq \|v\|_\infty, \quad \forall v \in H^1(\Omega) \cap L^\infty(\Omega),$$

can be established using density arguments in $H^1(\Omega)$ and the trace theorem. This ensures that $u_n \in H^1(\Omega) \cap L^\infty(\Gamma_1)$, and it is well known that

$$u_n \to u \quad \text{in} \quad H^1(\Omega).$$

2.2.1. Approximate Solutions

For a fixed $n \in \mathbb{N}$, we seek approximate solutions to problem (P_1), i.e., solutions of the finite-dimensional problem

$$u_n(t) = \sum_{j=1}^{n} g_{n_j} w_j, \quad \forall n \in \mathbb{N},$$

With the initial condition

$$u_n(0) = \sum_{j=1}^{n} g_{0n_j} w_j, \quad \forall n \in \mathbb{N},$$

such that

$$u_n(0) \to \varphi(x) \quad \text{in} \quad H^1(\Omega).$$

This leads to the problem

$$\begin{cases} ((u_n)_t, w_j) - (\Delta u_n, w_j) = 0, \quad j = 1, \ldots, n, \\ u_n(0) = u_{0n}. \end{cases} \tag{1}$$

By simplifying (1), we obtain

$$\begin{cases} ((u_n)_t, w_j) + (\nabla u_n, \nabla w_j) + \int_{\Gamma_1} |(u_n)_t|^{m-2} (u_n)_t w_j = \int_{\Gamma_1} h w_j, \quad j = 1, \ldots, n, \\ u_n(0) = u_{0n}. \end{cases} \tag{2}$$

Introducing

$$\begin{aligned} g_n &= (g_{n1}, \ldots, g_{nn})^T, \quad g_{0n} = (g_{0n1}, \ldots, g_{0nn})^T, \\ A_n &= ((\nabla w_i, \nabla w_j))_{i,j=1,\ldots,n}, \quad B_n = (w_1, \ldots, w_n)^T, \\ C_n(g) &= g + \int_{\Gamma_1} |B_n(x) \cdot g|^{m-2} B_n(x) \cdot g B_n(x)\, dx, \quad \forall g \in \mathbb{R}^n, \\ D_n(t) &= \int_{\Gamma_1} h B_n(x)\, dx. \end{aligned}$$

Thus, problem (2) can be rewritten in the following form:

$$\begin{cases} C_n\left((g_n)_t(t)\right) + A_n B_n(t) = D_n(t), \\ g_n(0) = g_{0n}. \end{cases} \tag{3}$$

Since the basis $(w_n)_n$ belongs to $H^1(\Omega) \cap L^m(\Gamma_1)$, it follows that $w_j \in L^m(\Gamma_1)$ for all $j = 1, \ldots, n$. Consequently, $|B_n| \in L^m(\Gamma_1)$ and C_n is continuous, satisfying

$$C_n = \nabla c_n, \quad \text{where} \quad c_n(g) = \frac{1}{2}|g|^2 + \frac{1}{m} \int_{\Gamma_1} |B_n(x) \cdot g|^m dx.$$

It can be easily shown that c_n is strictly convex because both terms in its definition are strictly convex. Furthermore, for any $g \neq 0$, we have

$$\lim_{k \to \infty} \frac{c_n(g)}{|g|} = \infty.$$

By applying the homeomorphism property, we reformulate problem (3) as

$$\begin{cases} (g_n)_t(t) = C_n^{-1}(-A_n B_n(t) + D_n(t)), \\ g_n(0) = g_{0n}. \end{cases} \tag{4}$$

Since

$$\left| C_n^{-1}(-A_n B_n(t) + D_n(t)) \right| \leq \|A_n\| |B_n| + |D_n|,$$

and using previously known results, we conclude that $D_n \in L^1(0, T)$. By applying Carathéodory's theorem, we obtain the existence of a local solution on $(0, t_k)$ for some $t_k > 0$.

2.2.2. Energy Estimates

Multiplying (2) by g_{nj} and summing over $j = 1, \ldots, n$, we derive

$$\begin{cases} ((u_n)_t, u_n) + (\nabla u_n, \nabla u_n) + \int_{\Gamma_1} |(u_n)_t|^{m-2} (u_n)_t u_n = \int_{\Gamma_1} h u_n, \\ u_n(0) = \varphi_n. \end{cases} \tag{5}$$

Applying Gronwall's inequality, we obtain

$$\frac{1}{\varepsilon}\|u_n\|_2^2 + \frac{2}{\varepsilon}\int_0^t \|\nabla u_n\|_2^2 + \frac{2}{\varepsilon}\int_0^t \int_{\Gamma_1} |(u_n)_t|^{m-2}(u_n)_t u_n \leq \left(\frac{1}{\varepsilon^2}\int_0^t \|h\|_2^2 + \frac{1}{\varepsilon}\|\varphi_n\|_2^2 \right) e^{\max_{n \in \mathbb{R}} t_n}. \tag{6}$$

Thus, the sequence (u_n) satisfies the necessary estimates to ensure the existence and uniqueness of weak solutions in $H^1(\Omega)$.

3. Main Results

This section presents fundamental results concerning the existence of solutions for the parabolic problem with nonlinear boundary conditions. We begin by establishing the existence of solutions to the linearized problem, which serves as a foundation for addressing the nonlinear case. The Faedo-Galerkin method is employed to construct approximate solutions, ensuring their convergence. Additionally, energy estimates are derived to analyze stability and regularity. These results provide a basis for further investigations into the qualitative behavior of solutions, including blow-up and global existence.

3.1. Existence of the Main Problem

This subsection is devoted to the study of a parabolic problem with nonlinear boundary conditions involving both damping and source terms. We consider the problem in a regular and bounded domain

$$Q = (0, T) \times \Omega, \quad \text{where } \Omega \subset \mathbb{R}^n, \quad T > 0.$$

The boundary of the domain is decomposed as

$$\partial\Omega = \Gamma_0 \cup \Gamma_1, \quad \Gamma_0 \cap \Gamma_1 = \emptyset.$$

We analyze the following problem:

$$\begin{cases} u_t - \Delta u = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,0) = \varphi(x), & x \in \Omega, \\ \frac{\partial u}{\partial \eta} = 0, & (x,t) \in \Gamma_0 \times (0,T), \\ \frac{\partial u}{\partial \eta} = -|u_t|^{q-2}u_t + u^p, & (x,t) \in \Gamma_1 \times (0,T). \end{cases} \tag{P$_p$}$$

Definition 3.1 A weak solution of the main problem (P$_p$) is a function u such that

1. $u \in L^\infty(0,T;H^1(\Omega))$ and $u_t \in L^2((0,T) \times \Gamma_1)$.
2. The trace of u on $(0,T) \times \partial\Omega$ has a distributional time derivative in $(0,T) \times \partial\Omega$, belonging to $L^q((0,T) \times \partial\Omega)$.
3. For all $\phi \in X$, where
$$X := \left\{ u \in H^1(\Omega) \mid u|_{\Gamma_1} \in L^q(\Gamma_1) \right\},$$
and for all $t \in [0,T]$, the following integral identity holds:
$$(u_t, \phi) + (\nabla u, \nabla \phi) + \int_{\Gamma_1} |u_t|^{q-2} u_t \phi \, dx = \int_{\Gamma_1} u^p \phi \, dx. \tag{7}$$
4. The initial condition is satisfied in the weak sense
$$u(0) = \varphi.$$

Lemma 3.2 Let $q > 1$ and $1 < p_0 < r$, where

$$r = \begin{cases} \frac{2(n-1)}{n-2}, & \text{if } n \geq 3, \\ \infty, & \text{if } n = 1,2. \end{cases} \tag{8}$$

Consider the space

$$\digamma := \left\{ u \in L^\infty \left(0,T;H^1(\Omega)\right), \quad u_t \in L^q\left((0,T) \times \Gamma_1\right) \right\}$$

endowed with the natural norm

$$\|u\|_\digamma^2 = \|u\|^2_{L^\infty(0,T;H^1(\Omega))} + \|u_t\|^2_{L^q((0,T)\times\Gamma_1)}. \tag{9}$$

Then, the embedding

$$\digamma \hookrightarrow C\left(0,T;L^{p_0}(\Gamma_1)\right)$$

is compact.

Proof 4 Let $y \in \digamma$. Then, by the trace theorem, we have

$$y \in L^\infty(0,T;L^{p_0}(\Gamma_1)).$$

For any $0 \leq s \leq t \leq T$, applying the interpolation inequality, we obtain

$$\|y(s) - y(t)\|_{p_0,\Gamma_1} \leq \|y(s) - y(t)\|^m_{1,\Gamma_1} \|y(s) - y(t)\|^{1-m}_{r,\Gamma_1}, \quad \forall m \in (0,1) \text{ such that } \frac{1}{p_0} = m + \frac{1-m}{r}.$$

Since $m > 1$, using Hölder's inequality twice, we estimate

$$\begin{aligned}\|y(s) - y(t)\|_{1,\Gamma_1} &\leq \int_s^t \|y_t\|_{1,\Gamma_1} \, dt \\ &\leq C(\Gamma_1) \int_s^t \|y_t\|_{q,\Gamma_1} \, dt \\ &\leq C(\Gamma_1)|t-s|^{\frac{1}{q'}} \|y_t\|_{L^q((0,T)\times\Gamma_1)} \,,\end{aligned}$$

for some positive constant $C(\Gamma_1)$. Then, by the trace-Sobolev embedding, we obtain

$$\|y(s) - y(t)\|_{1,\Gamma_1} \leq (C(\Gamma_1))^m \, |t-s|^{\frac{m}{q'}} \, \|y_t\|_{L^q((0,T)\times\Gamma_1)} \, \|\nabla y(s) - \nabla y(t)\|_2^{1-m} \,.$$

Thus, for a bounded sequence y_k in F, the sequence is equicontinuous in

$$C(0, T; L^{p_0}(\Gamma_1)).$$

Since the embedding $H^1(\Omega) \hookrightarrow L^{p_0}(\Gamma_1)$ is compact (using the standard partition-of-unity technique), the sequence $(y_k(t))_k$ is relatively compact in $L^{p_0}(\Gamma_1)$ for all $t \in [0, T]$. Applying Ascoli's theorem, we conclude that y_k is relatively compact in

$$C(0, T; L^{p_0}(\Gamma_1)).$$

This completes the proof.

Theorem 3.3 Let $q > 1$, $2 \leq p < r$, and $q > \frac{r}{r+1-p}$. Then, there exists a time $T > 0$ and a weak solution u of (P_p) in Q such that

$$u \in C(0, T; H^1(\Omega)), \quad u_t \in L^2((0,T)\times\Omega) \cap L^q((0,T)\times\Gamma_1). \tag{10}$$

Furthermore, the following energy identity holds for all $0 \leq s \leq t \leq T$:

$$\frac{1}{2} \, \|\nabla u\|_2^2\big|_s^t + \int_s^t \left(\|u_t\|_2^2 + \|u_t\|_{q,\Gamma_1}^q\right) dt = \int_s^t \int_{\Gamma_1} u^p u_t \, dx dt. \tag{11}$$

Moreover, the existence time T depends on the initial condition and parameters, given by

$$T = T\left(\|\varphi\|_{H^1(\Omega)}^2, q, p, \Omega, \Gamma_1\right),$$

and it is a decreasing function in the first variable.

Proof 5 We base our proof on the Schauder fixed-point theorem, which states "Let E be a locally convex topological vector space (LCTVS) such that any convex, closed, and bounded subset $A \subset E$ has a compact closure. If $u : A \to A$ is a continuous mapping, then u admits at least one fixed point." To apply this theorem, we define

$$X = \left\{u \in C(0, T; H^1(\Omega)), \quad u_t \in L^q((0,T)\times\Gamma_1), \quad u(0) = \varphi(x)\right\}, \tag{12}$$

$$B_R = \left\{u \in X \mid \|u\|_{L^\infty(0,T;H^1(\Omega))}^2 + \|u_t\|_{L^q((0,T)\times\Gamma_1)}^2 \leq R\right\}. \tag{13}$$

We define the mapping

$$\phi : X \to X, \quad v = \phi(u),$$

Where v is the solution of the following problem:

$$\begin{cases} v_t - \Delta v = 0, & (x,t) \in \Omega \times (0,T), \\ v(x,0) = \varphi(x), & x \in \Omega, \\ \frac{\partial v}{\partial \eta} = 0, & (x,t) \in \Gamma_0 \times (0,T), \\ \frac{\partial v}{\partial \eta} = -|v_t|^{q-2}v_t + u^p, & (x,t) \in \Gamma_1 \times (0,T). \end{cases} \tag{PP}$$

Since B_R is nonempty (as $\varphi \in B_R$ and satisfies $\|\varphi\|_{H^1(\Omega)} \leq R_0 \leq R$), and is closed and convex, we proceed to show that $\phi : B_R \to B_R$, ensuring 1. ϕ maps into itself for sufficiently large R and small T. 2. ϕ is continuous on B_R. 3. $\phi(B_R)$ is relatively compact in X.

• Step 1: Showing ϕ Maps Into Itself.

For $v \in B_R$, multiplying (PP) by v_t, integrating over Q, and using integration by parts, we obtain

$$\frac{1}{2}\|\nabla v\|_2^2 + \int_0^t \left(\|v_t\|_2^2 + \|v_t\|_{q,\Gamma_1}^q\right) = \int_0^t \int_{\Gamma_1} u^p v_t + \frac{1}{2}\|\nabla \varphi\|_2^2. \tag{14}$$

Applying Hölder's inequality to the right-hand side

$$\int_0^t \int_{\Gamma_1} u^p v_t + \frac{1}{2}\|\nabla \varphi\|_2^2 \leq \int_0^T \left(\int_{\Gamma_1} u^{q'p}\right)^{\frac{1}{q'}} \|v_t\|_{q,\Gamma_1}.$$

Since $\frac{qp}{q-1} < r$, using the trace-Sobolev embedding, we estimate

$$\int_0^t \int_{\Gamma_1} u^p v_t + \frac{1}{2}\|\nabla \varphi\|_2^2 \leq CR^p \int_0^T \|v_t\|_{q,\Gamma_1} + \frac{1}{2}\|\nabla \varphi\|_2^2.$$

Applying Young's inequality

$$\frac{1}{2}\|\nabla v\|_2^2 + \int_0^t \left(\|v_t\|_2^2 + \frac{1}{q'}\|v_t\|_{q,\Gamma_1}^q\right) \leq \frac{1}{2}R_0^2 + C^{q'}R^{q'p}T. \tag{15}$$

Thus, choosing R sufficiently large and T sufficiently small, we ensure $v \in B_R$, completing Step 1.

• Step 2: ϕ is Continuous on B_R.

For $u, \widetilde{u} \in B_R$, defining $w = v - \widetilde{v}$, we consider

$$\begin{cases} w_t - \Delta w = 0, & (x,t) \in \Omega \times (0,T), \\ w(x,0) = \varphi(x), & x \in \Omega, \\ \frac{\partial w}{\partial \eta} = 0, & (x,t) \in \Gamma_0 \times (0,T), \\ \frac{\partial w}{\partial \eta} = -|v_t|^{q-2}v_t + |\widetilde{v}_t|^{q-2}\widetilde{v}_t + u^p - \widetilde{u}^p, & (x,t) \in \Gamma_1 \times (0,T). \end{cases} \tag{PPP}$$

Following similar estimates as in Step 1, we show that $\phi(u) - \phi(\widetilde{u})$ vanishes as $u - \widetilde{u} \to 0$, proving continuity.

• Step 3: $\phi(B_R)$ is Relatively Compact in X.

By Lemma (8), the embedding $\digamma \hookrightarrow C(0,T; L^{p_0}(\Gamma_1))$ is compact, ensuring the existence of a Cauchy subsequence. By continuity, we conclude that $\phi(B_R)$ is relatively compact. Thus, applying Schauder's theorem, we obtain a fixed point $u = \phi(u)$, proving the existence of a weak solution.

3.2. Global Existence Results

In this subsection, we establish sufficient conditions ensuring the global existence of weak solutions to the problem (P_p). Using energy estimates and integral inequalities, we derive bounds that prevent finite-time blow-up and guarantee that solutions persist for all time. The key argument relies on controlling the nonlinear boundary terms and proving that the energy functional remains bounded over time. These results complement the local existence theory and provide insights into the long-term behavior of solutions.

Theorem 3.4 If $2 \leq p \leq m$ and $p < r$, then any weak solution obtained in Theorem 1 can be extended globally to $(0, \infty) \times \Omega$.

Proof 6 We start by using the following energy identity:

$$\|u_t\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + \|u_t\|_{q,\Gamma_1}^q = \frac{1}{p+1}\frac{d}{dt}\int_{\Gamma_1} u^{p+1}\,dx. \tag{16}$$

Rearranging, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}\right) = -\|u_t\|_2^2 - \|u_t\|_{q,\Gamma_1}^q \leq 0. \tag{17}$$

Now, we introduce the auxiliary functional

$$E(t) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{p+1}\|u\|_{p+1}^{p+1}.$$

Differentiating $E(t)$, we obtain

$$\begin{aligned}
E'(t) &= \frac{d}{dt}\left(\frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1} + \frac{2}{p+1}\|u\|_{p+1}^{p+1}\right) \\
&= \frac{d}{dt}\left(\frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}\right) + \frac{2}{p+1}\frac{d}{dt}\|u\|_{p+1}^{p+1} \\
&= -\|u_t\|_2^2 - \|u_t\|_{q,\Gamma_1}^q + 2\int_{\Gamma_1} u^p u_t\,dx \\
&\leq -\|u_t\|_{q,\Gamma_1}^q + \frac{1}{p+1}\|u\|_{p+1,\Gamma_1}^q + \frac{2^{p'}}{p'}\|u\|_{p+1,\Gamma_1}^{p+1} \\
&\leq Z\left(1 + \|u\|_{p+1,\Gamma_1}^{p+1}\right).
\end{aligned}$$

If $p = q + 1$, we obtain

$$E'(t) \leq Z(1 + E(t)).$$

Integrating over $(0, t)$, we get

$$E(t) \leq Z\int_0^T E(t)\,dt + ZT + E(0).$$

Applying Gronwall's lemma yields

$$E(t) \leq (ZT + E(0))\exp(ZT).$$

Thus, we conclude that

$$\|\nabla u\|_2, \quad \|u\|_{p+1,\Gamma_1} \in L^\infty(0, T_{\max}).$$

Moreover,

$$\begin{aligned}
\int_0^t \|u_t\|_2^2 &\leq \frac{1}{p+1}\|u\|_{p+1,\Gamma_1}^{p+1} - \frac{1}{2}\|\nabla u(0)\|_2^2, \\
&\leq \frac{1}{p+1}\|u\|_{L^\infty(0,T_{\max};L^{p+1}(\Gamma_1))}^{p+1} - \frac{1}{2}\|\nabla \varphi\|_2^2.
\end{aligned}$$

Now, using the integral representation

$$\begin{aligned}\|u\|_2^2 &= \left\|\varphi(x) + \int_0^t u_t\,dt\right\|_2^2 \\ &\leq \left(\|\varphi(x)\|_2 + \int_0^T \|u_t\|_2\right)^2 \\ &\leq 2\|\varphi(x)\|_2^2 + \int_0^T \|u_t\|_2^2 \\ &\leq 2\|\varphi(x)\|_2^2 + \frac{1}{p+1}\|u\|^{p+1}_{L^\infty(0,T_{\max};L^{p+1}(\Gamma_1))} - \frac{1}{2}\|\nabla\varphi\|_2^2.\end{aligned}$$

This leads to a contradiction, which proves the result.

Theorem 3.5 Suppose that $m > 1$, $2 \leq p < r$, and $m > \frac{r}{r+1-p}$, and that the measure of Γ_0 is positive. If the initial data satisfies $u_0 \in W$, where

$$W = \left\{u \in C([0,\infty); H^1(\Omega)), u_t \in L^m((0,\infty)\times\Gamma_1) \cap L^2((0,\infty)\times\Omega)\right\},$$

then there exists a global solution u of the problem on $(0,\infty)\times\Omega$, such that $u \in W$ for all $t \geq 0$. Furthermore, the energy identity holds for all $s, t \in (0,\infty)$

$$\int_s^t \|u_t\|_2^2 + \frac{1}{2}\left.\|\nabla u\|_2^2\right|_s^t + \int_s^t \|u_t\|_{q,\Gamma_1}^q = \frac{1}{p+1}\left.\int_{\Gamma_1} u^{p+1}\right|_s^t.$$

Proof 7 We first introduce an alternative and more explicit definition of

$$K(u) =$$

Which characterizes the set

$$W = \left\{\varphi \in H^1(\Omega) : K(\varphi) \geq 0, J(\varphi) < d\right\}.$$

Where

$$d = \inf_{\substack{u\in H^1(\Omega)\\ \frac{\partial u}{\partial v}\big|_{\Gamma_1}\neq 0}} \sup_{\mu>0} E(\mu u).$$

Since E is decreasing, we have

$$\begin{aligned}E(t) &= \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|^{p+1}_{p+1,\Gamma_1}, \\ E(t) &\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{B}{p+1}\|\nabla u\|^{p+1}_{H^1(\Omega)}, \\ E(t) + \|u\|^2_{H^1(\Omega)} &\geq \frac{1}{2}\|u\|^2_{H^1(\Omega)} - \frac{B}{p+1}\|u\|^{p+1}_{H^1(\Omega)}.\end{aligned}$$

Where

$$B = \sup_{\substack{u\in H^1(\Omega)\\ \frac{\partial u}{\partial v}\big|_{\Gamma_1}\neq 0}} \left(\frac{\|u\|_{H^1(\Omega)}}{\|u\|_{p+1,\Gamma_1}}\right).$$

Define the function

$$j(\|u\|_{H^1(\Omega)}) = \frac{1}{2}\|u\|^2_{H^1(\Omega)} - \frac{B}{p+1}\|u\|^{p+1}_{H^1(\Omega)}.$$

Then, we obtain the bound

$$j(\|u\|_{H^1(\Omega)}) \leq \left(\frac{1}{2} - \frac{1}{p+1}\right) B^{-\frac{2(p+1)}{p-1}}.$$

Moreover, since

$$d = \left(\frac{1}{2} - \frac{1}{p+1}\right) B^{-\frac{2(p+1)}{p-1}},$$

we obtain

$$E(t) + \|u\|^2_{H^1(\Omega)} \leq 2\left(\frac{1}{2} - \frac{1}{p+1}\right) B^{-\frac{2(p+1)}{p-1}}.$$

Since $E(t) < E(0)$, we deduce

$$E(0) + \|\varphi\|^2_{H^1(\Omega)} \leq d + B.$$

Additionally, we have

$$j(\|u\|_{H^1(\Omega)}) < E(t) + \|u\|^2_{H^1(\Omega)} < d + B.$$

Since $t \to \|u\|_{H^1(\Omega)}$ is continuous, we conclude that

$$\|u\|_{H^1(\Omega)} < B, \quad \forall t \in (0, T_{\max}).$$

Finally, using the embedding $H^1(\Omega) \hookrightarrow L^{p+1}(\Gamma_1)$, we establish

$$\|u\|_{p+1,\Gamma_1} \in L^\infty(0, T_{\max}),$$

Which concludes the proof.

4. Conclusion and Future Works

In this work, we investigated a modified heat equation with nonlinear damping Neumann boundary conditions, motivated by the heat transfer dynamics in the Columbia disaster. We established the local and global existence of solutions using energy estimates and compactness arguments, supported by numerical simulations. Future research can explore blow-up phenomena, optimal control strategies, stochastic effects, and advanced numerical methods to enhance model accuracy and applicability in real-world aerospace and thermal engineering problems.

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