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Fractional Approach to Two-Group Neutron Diffusion in Slab Reactors

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ABSTRACT

The two-energy neutron diffusion model in slab reactors characterizes neutron behavior across two energy groups: fast and thermal. Fast neutrons, generated by fission, decelerate through collisions, transitioning into thermal neutrons. This model employs diffusion equations to compute neutron flux distributions and reactor parameters, thereby optimizing reactor design and safety to ensure efficient neutron utilization and stable, sustained nuclear reactions. The primary objective of this research is to explore both analytical and numerical solutions to the two-energy neutron diffusion model in slab reactors. Specifically, we will utilize the Laplace transform method for an analytical solution of the two-energy neutron diffusion model. Subsequently, employing the Caputo differentiator, we transform the original neutron diffusion model into its fractional-order equivalents, yielding the fractional-order two-energy group neutron diffusion model in slab reactors. To address the resulting fractional-order system, we develop a novel approach aimed at reducing the 2β-order system to a β-order system, where $\beta \in (0, 1]$. This transformed system is then solved using the Modified Fractional Euler Method (MFEM), an advanced variation of the fractional Euler method. Finally, we present numerical simulations that validate our results and demonstrate their applicability.

1. Introduction

The nuclear reactor's complex fuel system competition is made up of reflectors, coolants, control rods, and other parts. The design and analysis of such reactors for various operating procedures is a complex undertaking that integrates several nuclear engineering disciplines. The distribution of neutron flux within the reactor core and the estimation of the mass and critical dimension are among the most crucial, this topic in the past decades has received great interest from the reactor physics community; for details, see [1]–[5] and the references therein. The nuclear fission reactor's neutron behaviour is described by the neutron diffusion equation, which is constructed from the neutron transport equation and simplified using Fick's law [3]. Therefore, finding a solution to the neutron

Vol. 5, No. 1, 2025, pp. 611–624

diffusion equation is important for understanding how neutrons behave in nuclear reactors. The homotopy perturbation method (HPM) was used to address the complex neutron diffusion equations [6], [7].

The neutron diffusion equation is one of the most important PDEs equations for expressing the behaviour of neutrons in nuclear reactors and a variety of physical applications [8]. In order to operate, nuclear reactors need to achieve criticality-a perfect balance between the production and loss of neutrons, the steady-state neutron transport equation, when the case is time independent, provides a mathematical expression for this equilibrium. Nonetheless, simplification is employed to facilitate practical analysis due to its complexity. As a suitable simplification, Fick's law establishes a relationship between the neutron flux and the neutron current. By resolving steady state, time-independent, and neutron diffusion equations, engineers can preserve criticality, guaranteeing a controlled and self-sustaining chain reaction inside the nuclear reactor; for additional details, see to [8]–[13].

The present research proposes sufficient analytical techniques based on the Laplace Transform Method (LTM) to offer a general solution for the integer-order two energy group neutron diffusion model in slab reactor. To do this, the slab radius r would be taken into account as a time domain. The Laplace Transform Method (LTM) is an effective strategy for solving neutron diffusion equations with multiple energy groups without the need for linearization, perturbation, or discretization. In addition, we focused in this research on study neutron diffusion model is transformed into its corresponding fractional-order equivalents that, producing the so-called fractional-order two energy group neutron diffusion model in the slab reactors by using the Caputo differentiator and developing a strategy to reduce a fractional-order system of order 2β into a replicated system of order β, where $\beta \in (0,1]$, see [14]–[20] to get a full overview about fractional calculus. Following the transformation of the 2β-order system into an β-order system, we solve the resulting system applying the Modified Fractional Euler Method (MFEM), a recent variation and a numerical modification of Fractional Euler Method (FEM); see [21]. At the end of this study, we present the numerical simulation that confirm our results and illustrate the applicability of the proposed methods using MATLAB techniques.

2. Preliminaries

This section focuses on presenting the basic definitions and theories related to fractional calculus, as well as definition and properties of the Laplace transform. However, to get a full overview about the fractional calculus and its applications, the reader may refer tho the references [22]–[56].

Definition 1 [57], [58] The Riemann-Liouville fractional integral of a function $g : [0, b] \to \mathbb{R}$ of order $\beta \in \mathbb{R}^+$ is given by:

$$J^\beta g(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x-t)^{\beta-1} g(t)dt, \;\; x \in [0,b], \tag{1}$$

With $\Gamma(\beta) = \int_0^\infty e^{-t} t^{\beta-1} dt$.

The following are some properties of the Riemann-Liouville integral [57], [59]:

1. $J^0 g(x) = g(x)$.
2. $J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} x^{\alpha+\beta}, \quad \alpha \geq 0, \, \beta \geq -1,$
3. $J^\alpha J^\beta g(x) = J^\beta J^\alpha g(x), \quad \alpha, \beta \geq 0.$
4. $J^\alpha J^\beta g(x) = J^{\alpha+\beta} g(x), \quad \alpha, \beta \geq 0.$

Definition 2 [57], [58] Let $\in \mathbb{C}^n([0,b])$ and $n-1 < \beta \leq n$ such that $n \in \mathbb{N}^*$. The Caputo fractional derivative of order β is defined by:

$${}^cD^\beta g(x) = J^{n-\beta} D^n g(x), = \frac{1}{\Gamma(n-\beta)} \int_0^x (x-t)^{n-\beta-1} g^{(n)}(t)dt. \tag{2}$$

Some properties of The Caputo fractional derivative are as follows [58], [59]:

1. ${}^cD^{\beta}a = 0, \quad a \in \mathbb{R}$.
2. ${}^cD^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, \ \alpha \geq 0, \ \beta > -1$.
3. ${}^cD^{\alpha}$ is a linear operator, i.e., for $\beta \geq 0$ and $\gamma, \lambda \in \mathbb{R}$, we have

$$ {}^cD^{\beta}(\gamma g(x) + \lambda h(x)) = \gamma{}^cD^{\alpha}g(x) + \lambda{}^cD^{\alpha}h(x). $$

4. For $n-1 < \beta \leq n$, $n \in \mathbb{N}$, we have:

$$ J^{\beta}\, {}^cD^{\beta}g(x) = g(x) - \sum_{i=1}^{n} \frac{x^i}{i!} f^i(0), \ \ x > 0. $$

Theorem 1 [60] **(Generalized Taylor's Theorem)** Assume that ${}^cD^{j\beta}g(x) \in \mathbb{C}(0,b]$, where $\beta \in (0,1]$ and $j = 0,1,2,\cdots,n+1$. Then, we can do the following to extend the function g concerning the node x_0:

$$ g(x) = \sum_{i=0}^{n} \frac{(x-x_0)^{i\beta}}{\Gamma(i\beta+1)} {}^cD^{i\beta}g(x_0) + \frac{(x-x_0)^{(n+1)\beta}}{\Gamma((n+1)\beta+1)} {}^cD^{(n+1)\beta}g(\xi), \tag{3} $$

$\forall x \in (0,b]$ *with* $\xi \in (0,x)$.

Definition 3 [61] Let function f be defined on $[0,\infty)$. Then, the Laplace transform $\mathscr{L}\{g\}$ is another function $G(x)$, which can be defined as:

$$ G(s) = \mathscr{L}\{g\} := \int_0^{\infty} e^{-sx} g(x)dx. \tag{4} $$

The Laplace transform has some of the following properties [61]:

1. $\mathscr{L}\{xy\} = -\frac{d}{ds}\mathscr{L}\{y\}$.
2. $\mathscr{L}\{g'(x)\} = -g(0) + s\mathscr{L}\{g\} = sG(s) - g(0)$.
3. $\mathscr{L}\{g''(x)\} = s^2G(s) - sg(0) - g'(0)$.

Now, we want to give a brief description of MFEM and show how it can be applied to solve the following initial value problems (for more details, see [21]):

$$ \begin{cases} {}^cD^{\beta}y(t) = g(t, y(t)), \ 0 < \beta \leq 1, \\ \\ \qquad\qquad y(0) = y_0. \end{cases} \tag{5} $$

In order to accomplish this, we suppose that $0 = t_0 < t_1 = t_0 + h < t_2 = t_0 + 2h < \cdots < t_n = t_0 + nh = b$ where the mesh points are $t_i = t_0 + ih$, $i = 1,2,\ldots,n$, with the step size $h = \frac{b}{n}$. Thus, by using Theorem 1's first three terms and making a few replacements, we get:

$$ u_0 = y_0, u_{i+1} = u_i + \frac{h^{\beta}}{\Gamma(\beta+1)} g\left(t_i + \frac{h^{\beta}}{2\Gamma(\beta+1)}, u_i + \frac{h^{\beta}}{2\Gamma(\beta+1)} g(t_i, u_i)\right), \tag{6} $$

Where u_i represents the numerical solution of problem (5), for $i = 1,2,\cdots,n-1$.

3. Two Energy Groups of Neutron Slab Reactor

In this section, we first want to treat the integer-order two energy group neutron diffusion model in slab reactors via LTM, and then this model will be addressed using MFEM in its fractional-order case.

3.1. Integer-Order Model

In this part, we will solve the integer-order two energy group neutron diffusion model in slab reactors using Laplace transform Method (LTM) and allowing the slap radius r to be a time domain. It is assumed for this reason that the integer-order two energy group neutron diffusion system have a single solution in the integration interval [1], [3], having the following form:

$$\begin{cases} \nabla^2\theta_1(r) + C_{11}\theta_1(r) + C_{12}\theta_2(r) = 0, \\ \nabla^2\theta_2(r) + C_{21}\theta_1(r) + C_{22}\theta_2(r) = 0, \end{cases} \tag{7}$$

Where C_{ii} is referred to as a group buckling and C_{ij} is a constant connection between fluxes in various energy groups of neutrons. Specifically, each of these constants can be defined by:

$$C_{ii} = \frac{x_i\nu_i \sum_{fi} - (\sum_{\gamma i} + \sum\sum_{sij})}{D_i}, C_{ij} = \frac{\sum\sum_{sij} + x_i\nu_j \sum_{fi}}{D_i}, D_i = \frac{1}{3(\sum_{fi} + \sum_{sii} + \sum\sum_{sij} + \sum_{\gamma i})}. \tag{8}$$

Equation (8) has determined the constants in terms of different macroscopic cross-sections, the number of neutrons generated by each fission for every group (ν_i), and the fraction of fission neutrons released with energies in the i^{th}-group (x_i). Actually, system (7) describes how neutrons behave in nuclear reactors, where each θ_i represents the neutron flux at a certain speed. Every flux reaches its maximum in the reactor's centre, its derivative disappears. Thus, the initial conditions could be written as:

$$\theta_i(0) = h_i, \ \theta_i'(0) = 0, \ i = 1, 2, \tag{9}$$

Where $h_i \in \mathbb{R}$ and the fluxes $\theta_i(r)$ are functions of independent variable r, for $i = 1, 2$. Throughout it is assumed that $\theta_i(r)$ are analytic functions for $r \geq 0$ and $i = 1, 2$.

In the content that follows, we solve two energy group neutron diffusion model in slab reactors by applying LTM and permitting the slab radius r be a time domain. To achieve this, let's go over the following information:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}. \tag{10}$$

As a result, system (7) can be reformulated as:

$$\begin{cases} \theta_1''(r) + C_{11}\theta_1(r) + C_{12}\theta_2(r) = 0, \\ \theta_2''(r) + C_{21}\theta_1(r) + C_{22}\theta_2(r) = 0, \end{cases} \tag{11}$$

With the initial conditions:

$$\begin{cases} \theta_i(0) = a_i, \ i = 1, 2, \\ \theta_i'(0) = b_i, \ i = 1, 2. \end{cases} \tag{12}$$

Applying the Laplace transform method (LTM) to both sides of (11), we find:

$$\begin{cases} \mathscr{L}(\theta_1''(r)) + C_{11}\mathscr{L}(\theta_1(r)) + C_{12}\mathscr{L}(\theta_2(r)) = 0, \\ \mathscr{L}(\theta_2''(r)) + C_{21}\mathscr{L}(\theta_1(r)) + C_{22}\mathscr{L}(\theta_2(r)) = 0, \end{cases} \tag{13}$$

Utilising the Laplace transform's properties, we can have

$$\begin{aligned} s^2\mathscr{L}\{\theta_1(r)\} - s\theta_1(0) - \theta_1'(0) + C_{11}\mathscr{L}\{\theta_1(r)\} \\ + C_{12}\mathscr{L}\{\theta_2(r) = 0, \\ s^2\mathscr{L}\{\theta_2(r)\} - s\theta_2(0) - \theta_2'(0) + C_{21}\mathscr{L}\{\theta_1(r)\} \\ + C_{22}\mathscr{L}\{\theta_2(r)\} = 0. \end{aligned} \tag{14}$$

By through supposing $\mathscr{L}\{\theta_i(r)\}) = G_i(s)$, for any $i = 1, 2$, we get:

$$\begin{cases} s^2G_1(s) - a_1 s - b_1 + C_{11}G_1(s) + C_{12}G_2(s) = 0, \\ s^2G_2(s) - a_2 s - b_2 + C_{21}G_1(s) + C_{22}G_2(s) = 0. \end{cases} \tag{15}$$

Thus, system (15) simplification yields:

$$\begin{cases} (s^2 + C_{11})G_1(s) + C_{12}G_2(s) = a_1 s + b_1, \\ C_{21}G_1(s) + (s^2 + C_{22})G_2(s) = a_2 s + b_2. \end{cases} \tag{16}$$

The system mentioned above can be represented in the matrix of the form:

$$\begin{bmatrix} s^2 + C_{11} & C_{12} \\ C_{21} & s^2 + C_{22} \end{bmatrix} \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} s + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \tag{17}$$

or

$$M(s)G(s) = As + B, \tag{18}$$

With

$$M(s) = \begin{bmatrix} -s^2 + C_{11} & C_{12} \\ C_{21} & s^2 + C_{22} \end{bmatrix}, G(s) = \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix}, \tag{19}$$

and

$$A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \tag{20}$$

So, if we assume that $M(s)$ is invertible, we can obtain:

$$G(s) = M^{-1}(s)\left(As + B\right). \tag{21}$$

With a prepared MATLAB code and the knowledge that $\theta_i(r) = \mathscr{L}^{-1}\{G_i(s)\}$ for all $i = 1, 2$, system (21) may therefore be solved numerically. This would give the solution $\theta_i(r)$ of system (11).

3.2. Fractional-Order Model

$$\begin{cases} {}^cD^{2\beta}\theta_1(r) + C_{11}\theta_1(r) + C_{12}\theta_2(r) = 0, \\ {}^cD^{2\beta}\theta_2(r) + C_{21}\theta_1(r) + C_{22}\theta_2(r) = 0, \end{cases} \tag{22}$$

With the following initial conditions:

$$\begin{cases} \theta_i(0) = a_i, \ i = 1, 2, \\ \theta_i'(0) = b_i, \ i = 1, 2. \end{cases} \tag{23}$$

It is clear that system (22) is of order 2β. For this reason, we need to develop a manner that can deal with such a system with such a fractional-order. To do so, we introduce the following lemma that aims to reduce the system of 2β-order into another duplicated system of β-order, where $0 < \beta \leq 1$.

Lemma 1 For any fractional differential equations of order $n\beta, n \in \mathbb{Z}^+$ and $0 < \beta \leq 1$, with functions possessing values in $\mathbb{R}$ can be transformed into a system of fractional differential equations of order β with values in $\mathbb{R}^{nd}$.

Proof 1 In order to demonstrate this result, we first have to consider the scaler case that takes place whenever $d = 1$, and then we'll think about the last case that holds when $d > 1$. Because of this, we should be aware that in its scaler case, the general form of the fractional differential equations of fractional-order $n\beta$ can be provided by:

$${}^cD^{n\beta}y(t) = F(t, y(t), {}^cD^{\beta}y(t), {}^cD^{2\beta}y(t), \cdots, {}^cD^{(n-1)\beta}y(t)) \tag{24}$$

Where F is a continuous function defined on the subset $I \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, so that it takes values in $\mathbb{R}$ for a given intend I. Now, we define:

$$\mathbf{\Psi}(t, v_0, v_1, \cdots, v_{n-1}) = (v_1, v_2, \cdots, F(t, v_0, v_1, \cdots, v_{n-1})) \tag{25}$$

as a continuous function on $I \times \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ as F, but it takes the values in $\mathbb{R}^n$. Regarding this, we take into consideration the following formula:

$$^cD^\beta \mathbf{Y}(t) = \mathbf{\Psi}(t, \mathbf{Y}(t)), \textit{ for } t \in I. \tag{26}$$

Next, we wish to prove that $x : I \to \mathbb{R}$ is a solution of equation (25) if and only if the function

$$\begin{aligned} \mathbf{X} :& I \to \mathbb{R}^n, \\ & t \to (x(t), {}^cD^\beta x(t), {}^cD^{2\beta} x(t), \cdots, {}^cD^{(n-1)\beta} x(t)). \end{aligned} \tag{27}$$

In order to achieve this, we suppose that $\mathbf{X}$ is a solution to equation (25) such that $\mathbf{X}$ is defined above. Then we have:

$$\begin{aligned} ^cD^\beta \mathbf{X}(t) &= \begin{pmatrix} ^cD^\beta x(t) \\ ^cD^{2\beta} x(t) \\ \vdots \\ ^cD^{(n-1)\beta} x(t) \\ ^cD^{n\beta} x(t) \end{pmatrix} \\ &= \begin{pmatrix} ^cD^\beta x(t) \\ ^cD^{2\beta} x(t) \\ \vdots \\ ^cD^{(n-1)\beta} x(t) \\ F(t, x(t), {}^cD^\beta x(t), {}^cD^{2\beta} x(t), \cdots, {}^cD^{(n-1)\beta} x(t)) \end{pmatrix} \\ &= \mathbf{\Psi}(t, \mathbf{X}(t)). \end{aligned} \tag{28}$$

Herein, the converse of the above discussion is similar. Now, for the case of $d > 1$, the above proof can be re-read again, replacing each occurrence of $\mathbb{R}$ with $\mathbb{R}^d$ to obtain the desired result.

Considering Lemma 1, we transform system (22), which is of order 2β, into the equivalent fractional system of order β, where $0 < \beta \leq 1$. In order to accomplish this, we can be rewrite system (22) as follow:

$$\begin{cases} ^cD^{2\beta}\theta_1(r) = -\Big(C_{11}\theta_1(r) + C_{12}\theta_2(r)\Big), \\ ^cD^{2\beta}\theta_2(r) = -\Big(C_{21}\theta_1(r) + C_{22}\theta_2(r)\Big), \end{cases} \tag{29}$$

Now, we consider the following hypotheses:

$$\begin{aligned} g_1\left(r, \theta_1(r), {}^cD^\beta\theta_1(r), \theta_2(r), {}^cD^\beta\theta_2(r)\right) &= -\Big(C_{11}\theta_1(r) + C_{12}\theta_2(r)\Big), \\ g_2\left(r, \theta_1(r), {}^cD^\beta\theta_1(r), \theta_2(r), {}^cD^\beta\theta_2(r)\right) &= -\Big(C_{21}\theta_1(r) + C_{22}\theta_2(r)\Big). \end{aligned} \tag{30}$$

Next, system (30) converts into:

$$\begin{aligned} ^cD^{2\beta}\theta_1(r) &= g_1\left(r, \theta_1(r), {}^cD^\beta\theta_1(r), \theta_2(r), {}^cD^\beta\theta_2(r)\right), \\ ^cD^{2\beta}\theta_2(r) &= g_2\left(r, \theta_1(r), {}^cD^\beta\theta_1(r), \theta_2(r), {}^cD^\beta\theta_2(r)\right). \end{aligned} \tag{31}$$

Let $v_i(r) =^c D^\beta \theta_i(r)$ for $i = 1, 2$. Then we get

$$\begin{aligned} {}^cD^\beta \theta_1(r) &= v_1(r) = f_1\left(r, \theta_1(r), v_1(r), \theta_2(r), v_2(r)\right), \\ {}^cD^\beta v_1(r) &=^c D^{2\beta} \theta_1(r) = g_1\left(r, \theta_1(r), v_1(r), \theta_2(r), v_2(r)\right), \\ {}^cD^\beta \theta_2(r) &= v_2(r) = f_2\left(r, \theta_1(r), v_1(r), \theta_2(r), v_2(r)\right), \\ {}^cD^\beta v_2(r) &=^c D^{2\beta} \theta_2(r) = g_2\left(r, \theta_1(r), v_1(r), \theta_2(r), v_2(r)\right), \end{aligned} \tag{32}$$

With the initial conditions:

$$\theta_i(0) = a_i, \; v_i(0) = b_i, \text{ for } \; i = 1, 2. \tag{33}$$

In the following, we want here to take the slab radius r into consideration as a time domain. Consequently, to solve the transformed system (32) using MFEM [21], we subdivide the interval $I = [0, b]$ as $0 = r_0 < r_1 = r_0 + h < r_2 = r_0 + 2h < \cdots < r_n = r_0 + nh = b$ such that $r_i = r_0 + ih$ and $h = \frac{b}{n}$, for $i = 1, 2$. To keep things simple, we indicate respectively $f_i(r, \theta_1(r), v_1(r), \theta_2(r), v_2(r))$ and $g_i(r, \theta_1(r), v_1(r), \theta_2(r), v_2(r))$ by $f_i(\Omega)$ and $g_i(\Omega)$, where $\Omega = (r, \theta_1(r), v_1(r), \theta_2(r), v_2(r))$, for $i = 1, 2$. The following states can now be obtained applying the MFEM's main formula (6):

$$\begin{aligned} \theta_1(r_{i+1}) = \theta_1(r_i) + \frac{h^\beta}{\Gamma(\beta+1)} f_1\Bigg(r_i + \frac{h^\beta}{2\Gamma(\beta+1)}, \\ \theta_1(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} f_1(\Omega), \theta_2(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} f_2(\Omega), \\ v_1(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} g_1(\Omega), v_2(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} g_2(\Omega)\Bigg), \\ v_1(r_{i+1}) = v_1(r_i) + \frac{h^\beta}{\Gamma(\beta+1)} g_1\Bigg(r_i + \frac{h^\beta}{2\Gamma(\beta+1)}, \\ \theta_1(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} f_1(\Omega), \theta_2(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} f_2(\Omega), \\ v_1(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} g_1(\Omega), v_2(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} g_2(\Omega)\Bigg), \\ \theta_2(r_{i+1}) = \theta_2(r_i) + \frac{h^\beta}{\Gamma(\beta+1)} f_2\Bigg(r_i + \frac{h^\beta}{2\Gamma(\beta+1)}, \\ \theta_1(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} f_1(\Omega), \theta_2(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} f_2(\Omega), \\ v_1(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} g_1(\Omega), v_2(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} g_2(\Omega)\Bigg), \\ v_2(r_{i+1}) = v_2(r_i) + \frac{h^\beta}{\Gamma(\beta+1)} g_2\Bigg(r_i + \frac{h^\beta}{2\Gamma(\beta+1)}, \\ \theta_1(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} f_1(\Omega), \theta_2(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} f_2(\Omega), \\ v_1(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} g_1(\Omega), v_2(r_i) + \frac{h^\beta}{2\Gamma(\beta+1)} g_2(\Omega)\Bigg), \end{aligned} \tag{34}$$

for $i = 1, 2$. In actuality, system (34) represents an approximation solution of system (32), and as a result, the approximation solution of system (22) is represented by $(\theta_1(t), \theta_2(t))$.

4. Numerical Simulations

This section aims to examine the effects of boundary conditions on the numerical results of slap reactor simulations, namely the extrapolated boundary condition and the zero flux boundary condition. We compare the results generated by different boundary conditions in an attempt to discover the impact these choices have on the reliability and precision of the simulation results. In the following, we use the LTM and MFEM, respectively, to explain and illustrate the numerical solutions of the integer- and fractional-order two energy group neutron diffusion model in slab reactors. For this reason, we list in the following values of the parameters in Table 1 and Table 2 obtained from the reference [62].

Table 1. Two energy group data

Fast Energy Group		
$\sum_{f_1} = 0.0010484cm^{-1}$	$\sum_{\gamma_1} = 0.0010046cm^{-1}$	$\sum_{S11} = 0.62568cm^{-1}$
$\sum_{S12} = 0.029227cm^{-1}$	$v_1 = 2.5$	$\chi_1 = 1.0$
Thermal Energy Group		
$\sum_{f_2} = 0.05063cm^{-1}$	$\sum_{\gamma_2} = 0.025788cm^{-1}$	$\sum_{S22} = 2.443838cm^{-1}$
$\sum_{S21} = 0.00000cm^{-1}$	$v_2 = 2.5$	$\chi_2 = 0.0$

Table 2. The values of the coefficients C_{ij} are calculated from equation (8)

C_{ij}	$i = 1$	$j = 2$
$i = 1$	-0.0564834	0.220978
$j = 2$	0.249474	-0.577793

The two-energy group of neutrons reactors diffusions system related to the slab reactor can be rewritten in its classical case as:

$$\begin{cases} \theta_1''(r) + C_{11}\theta_1(r) + C_{12}\theta_2(r) = 0, \\ \theta_2''(r) + C_{21}\theta_1(r) + C_{22}\theta_2(r) = 0, \end{cases} \tag{35}$$

With the following initial conditions:

$$\theta_1(0) = 2.766976,\ \theta_2(0) = 1,\ \theta_1'(0) = 1,\ \theta_2'(0) = 0. \tag{36}$$

On the other hand, the fractional order of the two-energy groups of neutrons reactors diffusions system related to the slab reactor can be rewritten in the following form:

$$\begin{cases} {}^cD^{2\beta}\theta_1(r) + C_{11}\theta_1(r) + C_{12}\theta_2(r) = 0, \\ {}^cD^{2\beta}\theta_2(r) + C_{21}\theta_1(r) + C_{22}\theta_2(r) = 0, \end{cases} \tag{37}$$

With the following initial conditions:

$$\theta_1(0) = 2.766976,\ \theta_2(0) = 1,\ {}^cD^{\beta}\theta_1(0) = 0,\ {}^cD^{\beta}\theta_2(0) = 0. \tag{38}$$

Now, in order to confirm that the fractionalization technique implemented by Lemma 1, we present a numerical comparison in Fig. 1 between the MFEM's solution of system (22) and the LTM's solution of system (11). Given that figure, it is evident to see that the two solutions perfectly correspond. Consequently, using MATLAB, the two energy groups of the neutron reactor diffusion model is effectively simulated when $\beta = 1$.

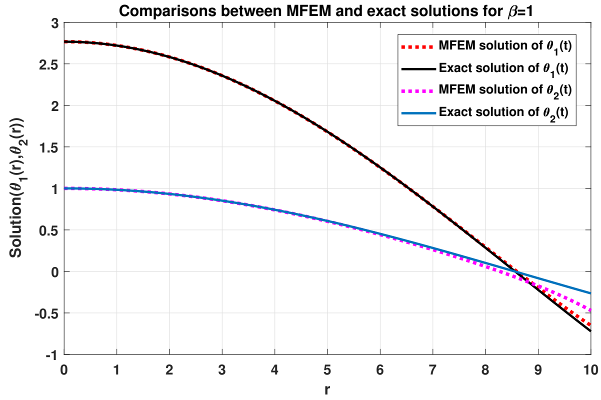

Fig. 1. LTM and MFEM solutions for two-energy groups of neutrons reactors diffusions system

We plot in Fig. 2 and Fig. 3 respectively the numerical solutions for $\theta_1(r)$ and $\theta_2(r)$ which are performed by MFEM's according to different values of β.

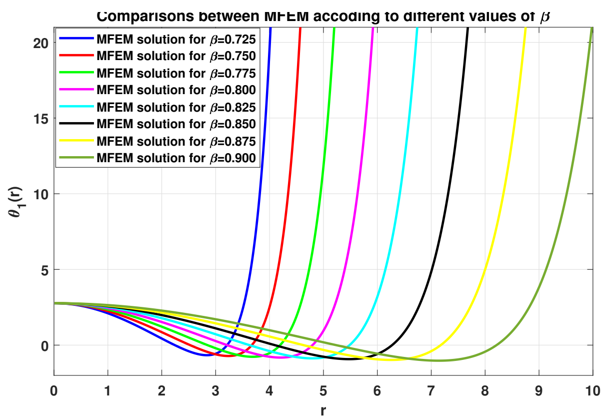

Fig. 2. MFEM's numerical solutions for $\theta_1(r)$ according to different values of β

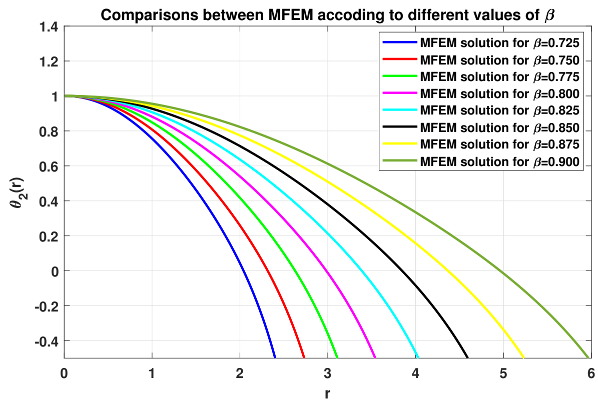

Fig. 3. MFEM's numerical solutions for $\theta_2(r)$ according to different values of β

In Fig. 4 and Fig. 5, we present respectively two numerical comparisons for $\theta_1(r)$ and $\theta_2(r)$, which are carried between LTM's solutions and several MFEM's solutions according to different values of β.

Vol. 5, No. 1, 2025, pp. 611–624

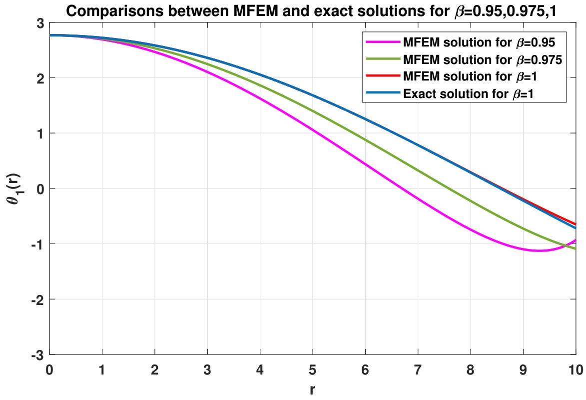

Fig. 4. LTM's and MFEM's solutions for $\theta_1(r)$ when $\beta = 0.95, 0.975, 1$

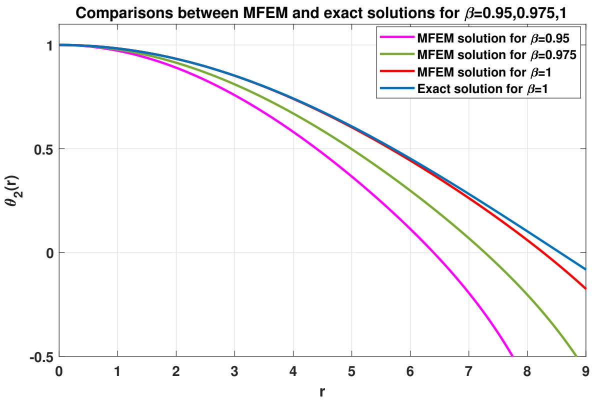

Fig. 5. LTM's and MFEM's solutions for $\theta_2(r)$ when $\beta = 0.95, 0.975, 1$

5. Conclusion

In this research, we have successfully proposed two effective approaches to solve the integer-order and fractional-order two-energy group neutron diffusion models in slab reactors. The Laplace transform approach was employed to handle the integer-order model, demonstrating its capability in providing precise analytical solutions. Concurrently, the modified fractional Euler method was utilized to address the fractional-order model, showcasing its robustness in dealing with the complexities introduced by fractional calculus. Our results have been validated through extensive numerical comparisons, confirming the accuracy and effectiveness of both methods.

The findings of this study not only contribute to the existing body of knowledge in neutron diffusion modeling but also open up several avenues for future research. One potential direction is the exploration of other fractional-order numerical methods, such as the fractional Adams-Bashforth-Moulton method, which could offer improved accuracy or computational efficiency. Additionally, extending the analysis to more complex reactor geometries, such as cylindrical or spherical reactors, would be valuable in understanding the applicability and limitations of these methods in different contexts.

Another important area for future investigation is the impact of varying boundary conditions, such as reflective or periodic boundaries, on the performance of the proposed methods. Such studies could provide deeper insights into the stability and convergence properties of the solutions under different physical scenarios. Furthermore, the integration of stochastic elements into the fractional-order models could be explored to better simulate the inherent uncertainties in reactor behavior.

Finally, future work could focus on the practical implementation of these methods in real-world reactor design and safety analysis. This includes the development of more sophisticated computational tools that can handle large-scale simulations, as well as collaborations with industry to apply these models in operational settings. By addressing these future challenges, we aim to further refine and expand the applicability of our approaches, ultimately contributing to the advancement of neutron diffusion modeling in nuclear engineering.

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